

# TORIFICATION AND FACTORIZATION OF BIRATIONAL MAPS

DAN ABRAMOVICH, KALLE KARU, KENJI MATSUKI, AND JAROSŁAW WŁODARCZYK

ABSTRACT. Building on work of the fourth author in [69], we prove the weak factorization conjecture for birational maps in characteristic zero: a birational map between complete nonsingular varieties over an algebraically closed field  $K$  of characteristic zero is a composite of blowings up and blowings down with smooth centers.

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## 0. INTRODUCTION

We work over an algebraically closed field  $K$  of characteristic 0. We denote the multiplicative group of  $K$  by  $K^*$ .

**0.1. Statement of the main result.** The purpose of this paper is to give a proof for the following weak factorization conjecture of birational maps. We note that another proof of this theorem was given by the fourth author in [70]. See section 0.12 for a brief comparison of the two approaches.

**Theorem 0.1.1** (Weak Factorization). *Let  $\phi : X_1 \dashrightarrow X_2$  be a birational map between complete nonsingular algebraic varieties  $X_1$  and  $X_2$  over an algebraically closed field  $K$  of characteristic zero, and let  $U \subset X_1$  be an open set where  $\phi$  is an isomorphism. Then  $\phi$  can be factored into a sequence of blowings up and blowings down with smooth irreducible centers disjoint from  $U$ , namely, there exists a sequence of birational maps between complete nonsingular algebraic varieties*

$$X_1 = V_0 \xrightarrow{\varphi_1} V_1 \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_i} V_i \xrightarrow{\varphi_{i+1}} V_{i+1} \xrightarrow{\varphi_{i+2}} \cdots \xrightarrow{\varphi_{l-1}} V_{l-1} \xrightarrow{\varphi_l} V_l = X_2$$

where

1.  $\phi = \varphi_l \circ \varphi_{l-1} \circ \cdots \circ \varphi_2 \circ \varphi_1$ ,
2.  $\varphi_i$  are isomorphisms on  $U$ , and
3. either  $\varphi_i : V_i \dashrightarrow V_{i+1}$  or  $\varphi_i^{-1} : V_{i+1} \dashrightarrow V_i$  is a morphism obtained by blowing up a smooth irreducible center disjoint from  $U$ .

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Furthermore, there is an index  $i_0$  such that for all  $i \leq i_0$  the map  $V_i \dashrightarrow X_1$  is a projective morphism, and for all  $i \geq i_0$  the map  $V_i \dashrightarrow X_2$  is a projective morphism. In particular, if  $X_1$  and  $X_2$  are projective then all the  $V_i$  are projective.

**0.2. Strong factorization.** If we insist in the assertion above that  $\varphi_1^{-1}, \dots, \varphi_{i_0}^{-1}$  and  $\varphi_{i_0+1}, \dots, \varphi_l$  be regular maps for some  $i_0$ , we obtain the following strong factorization conjecture.

**Conjecture 0.2.1** (Strong Factorization). *Let the situation be as in Theorem 0.1.1. Then there exists a diagram*

$$\begin{array}{ccc} & Y & \\ \psi_1 \swarrow & & \searrow \psi_2 \\ X_1 & \xrightarrow{\phi} & X_2 \end{array}$$

where the morphisms  $\psi_1$  and  $\psi_2$  are composites of blowings up of smooth centers disjoint from  $U$ .

See Section 6.1 for further discussion.

**0.3. Generalizations of the main theorem.** We consider the following categories, in which we denote the morphisms by “broken arrows”:

1. the objects are complete nonsingular algebraic spaces over an arbitrary field  $L$  of characteristic 0, and broken arrows  $X \dashrightarrow Y$  denote birational  $L$ -maps, and
2. the objects are compact complex manifolds, and broken arrows  $X \dashrightarrow Y$  denote bimeromorphic maps.

Given two broken arrows  $\phi : X \dashrightarrow Y$  and  $\phi' : X' \dashrightarrow Y'$  we define an *absolute isomorphism*  $g : \phi \rightarrow \phi'$  as follows:

- In case  $X$  and  $Y$  are algebraic spaces over  $L$ , and  $X', Y'$  are over  $L'$ , then  $g$  consists of an isomorphism  $\sigma : \text{Spec } L \rightarrow \text{Spec } L'$ , together with a pair of biregular  $\sigma$ -isomorphisms  $g_X : X \rightarrow X'$  and  $g_Y : Y \rightarrow Y'$ , such that  $\phi' \circ g_X = g_Y \circ \phi$ .
- In the analytic case,  $g$  simply consists of a pair of biregular isomorphisms  $g_X : X \rightarrow X'$  and  $g_Y : Y \rightarrow Y'$ , such that  $\phi' \circ g_X = g_Y \circ \phi$ .

**Theorem 0.3.1.** *Let  $\phi : X_1 \dashrightarrow X_2$  be as in case (1) or (2) above. Let  $U \subset X_1$  be an open set where  $\phi$  is an isomorphism. Then  $\phi$  can be factored, functorially with respect to absolute isomorphisms, into a sequence of blowings up and blowings down with smooth centers disjoint from  $U$ . Namely, to any such  $\phi$  we associate a diagram in the corresponding category*

$$X_1 = V_0 \xrightarrow{\varphi_1} V_1 \xrightarrow{\varphi_2} \dots \xrightarrow{\varphi_i} V_i \xrightarrow{\varphi_{i+1}} V_{i+1} \xrightarrow{\varphi_{i+2}} \dots \xrightarrow{\varphi_{l-1}} V_{l-1} \xrightarrow{\varphi_l} V_l = X_2$$

where

1.  $\phi = \varphi_l \circ \varphi_{l-1} \circ \dots \circ \varphi_2 \circ \varphi_1$ ,
2.  $\varphi_i$  are isomorphisms on  $U$ , and
3. either  $\varphi_i : V_i \dashrightarrow V_{i+1}$  or  $\varphi_i^{-1} : V_{i+1} \dashrightarrow V_i$  is a morphism obtained by blowing up a smooth center disjoint from  $U$ .
4. *Functoriality:* if  $g : \phi \rightarrow \phi'$  is an absolute isomorphism, carrying  $U$  to  $U'$ , and  $\varphi'_i : V'_i \dashrightarrow V'_{i+1}$  is the factorization of  $\phi'$ , then the resulting rational maps  $g_i : V_i \dashrightarrow V'_i$  are biregular.
5. Moreover, there is an index  $i_0$  such that for all  $i \leq i_0$  the map  $V_i \dashrightarrow X_1$  is a projective morphism, and for all  $i \geq i_0$  the map  $V_i \dashrightarrow X_2$  is a projective morphism.
6. Let  $E_i \subset V_i$  be the exceptional divisor of  $V_i \rightarrow X_1$  (respectively,  $V_i \rightarrow X_2$ ) in case  $i \leq i_0$  (respectively,  $i \geq i_0$ ). Then the above centers of blowings up in  $V_i$  have normal crossings with  $E_i$ . If, moreover,  $X_1 \setminus U$  (respectively,  $X_2 \setminus U$ ) is a normal crossings divisor, then the centers of blowing up have normal crossings with the inverse images of this divisor.

*Remarks.* 1. Note that, in order to achieve functoriality, we cannot require the centers of blowing up to be irreducible.

2. Functoriality implies, as immediate corollaries, the existence of factorization over any field of characteristic 0, as well as factorization, equivariant under the action of a group  $G$ , of a  $G$ -equivariant birational map.

3. The same theorem holds true for varieties or algebraic spaces of dimension  $d$  over a perfect field of characteristic  $p > 0$  assuming that canonical embedded resolution of singularities holds true for varieties or algebraic spaces of dimension  $d + 1$  in characteristic  $p$ . The proof for varieties goes through word for word as in this paper, while for the algebraic space case one needs to recast some of our steps from the Zariski topology to the étale topology (see [30], [40]).
4. While this theorem clearly implies the main theorem as a special case, we prefer to carry out the proof of the main theorem throughout the text, and indicating the changes one needs to perform for proving Theorem 0.3.1 in section 5.
5. This is by no means the most general case to which our methods apply, and we are aware of some applications which are not covered in this statement. It may be of interest in the future to codify a minimal set of axioms needed to carry out this line of proof of weak factorization.

**0.4. Early origins of the problem.** The history of the factorization problem of birational maps could be traced back to the Italian school of algebraic geometers, who already knew that the operation of blowing up points on surfaces is a fundamental source of richness for surface geometry: the importance of the strong factorization theorem in dimension 2 (see [71]) cannot be overestimated in the analysis of the birational geometry of algebraic surfaces. We can only guess that Zariski, possibly even members of the Italian school, contemplated the problem in higher dimension early on, but refrained from stating it before results on resolution of singularities were available. The question of strong factorization was explicitly stated by Hironaka as “Question (F’)” in [23], Chapter 0, §6, and the question of weak factorization was raised in [49]. The problem remained largely open in higher dimensions despite the efforts and interesting results of many (see e.g. Crauder [11], Kulikov [37], Moishezon [43], Schaps [60], Teicher [64]). Many of these were summarized by Pinkham [52], where the weak factorization conjecture is explicitly stated.

**0.5. The toric case.** For toric birational maps, the equivariant versions of the weak and strong factorization conjectures were posed in [49] and came to be known as Oda’s weak and strong conjectures. While the toric version can be viewed as a special case of the general factorization conjectures, many of the examples demonstrating the difficulties in higher dimensions are in fact toric (see Hironaka [22], Sally [58], Shannon [61]). Thus Oda’s conjecture presented a substantial challenge and combinatorial difficulty. In dimension 3, Danilov’s proof of Oda’s weak conjecture [16] was later supplemented by Ewald [18]. Oda’s weak conjecture was solved in arbitrary dimension by J. Włodarczyk in [68], and another proof was given by R. Morelli in [44] (see also [45], [3]). An important combinatorial notion which Morelli introduced into this study is that of a *cobordism* between fans. The algebro-geometric realization of Morelli’s combinatorial cobordism is the notion of a *birational cobordism* introduced in [69].

In [44], R. Morelli also proposed a proof of Oda’s *strong* conjecture. A gap in this proof, which was not noticed in [3], was recently discovered by K. Karu. As far as we know, Oda’s strong conjecture stands unproven at present even in dimension 3.

**0.6. The local version.** There is a local version of the factorization conjecture, formulated and proved in dimension 2 by Abhyankar ([5], Theorem 3). Christensen [9] posed the problem in general and solved it for some special cases in dimension 3. Here the varieties  $X_1$  and  $X_2$  are replaced by appropriate birational local rings dominated by a fixed valuation, and blowings up are replaced by monoidal transforms subordinate to the valuation. The weak form of this local conjecture was recently solved by S. D. Cutkosky in a series of papers [12, 13]. Cutkosky also shows that the strong version of the conjecture follows from Oda’s strong factorization conjecture for toric morphisms. In a sense, Cutkosky’s result says that the only *local* obstructions to solving the global strong factorization conjecture lie in the toric case.

**0.7. Birational cobordisms.** Our method is based upon the theory of birational cobordisms [69]. As mentioned above, this theory was inspired by the combinatorial notion of polyhedral cobordisms of R. Morelli [44], which was used in his proof of weak factorization for toric birational maps.

Given a birational map  $\phi : X_1 \dashrightarrow X_2$ , a *birational cobordism*  $B_\phi(X_1, X_2)$  is a variety of dimension  $\dim(X_1) + 1$  with an action of the multiplicative group  $K^*$ . It is analogous to the usual cobordism  $B(M_1, M_2)$  between differentiable manifolds  $M_1$  and  $M_2$  given by a Morse function  $f$ . In the differential setting one can construct an action of the additive real group  $\mathbb{R}$ , where the “time”  $t \in \mathbb{R}$  acts as a diffeomorphism induced by integrating the vector field  $\text{grad}(f)$ ; hence the multiplicative group  $(\mathbb{R}_{>0}, \times) = \exp(\mathbb{R}, +)$  acts as well. The critical points of  $f$  are precisely the fixed points of the action of the multiplicative group, and the homotopy type of fibers of  $f$  changes when we pass through these critical points. Analogously, in the

algebraic setting “passing through” the fixed points of the  $K^*$ -action induces a birational transformation. Looking at the action on the tangent space at each fixed point, we obtain a locally toric description of the transformation. This already gives the main result of [69]: a factorization of  $\phi$  into *locally toric birational maps* among varieties with locally toric structures. Such birational transformations can also be interpreted using the work of Brion-Procesi, Thaddeus, Dolgachev-Hu and others (see [8, 65, 66, 17]), which describes the change of Geometric Invariant Theory quotient associated to a change of linearization.

**0.8. Locally toric versus toroidal structures.** Considering the fact that weak factorization has been proven for *toroidal* birational maps ([68], [44], [3]), one might naïvely think that a locally toric factorization, as indicated in the previous paragraph, would already provide a proof for Theorem 0.1.1.

However, in the locally toric structure obtained from a cobordism, the embedded tori chosen may vary from point to point, while a toroidal structure (see below) requires the embedded tori to be induced from one fixed open set. Thus there is still a gap between the notion of locally toric birational maps and that of toroidal birational maps.

**0.9. Torification.** In order to bridge over this gap, we follow ideas introduced by Abramovich and De Jong in [1], and blow up suitable open subsets, called *quasi-elementary cobordisms*, of the birational cobordism  $B_\phi(X_1, X_2)$  along *toric ideals*. This operation induces a toroidal structure in a neighborhood of each connected component  $F$  of the fixed point set, on which the action of  $K^*$  is a *toroidal action* (we say that the blowing up *torifies* the action of  $K^*$ ). Now the birational transformation “passing through  $F$ ” is toroidal. We use canonical resolution of singularities to desingularize the resulting varieties, bringing ourselves to a situation where we can apply the factorization theorem for toroidal birational maps. This completes the proof of Theorem 0.1.1.

**0.10. Relation with the minimal model program.** It is worthwhile to note the relation of the factorization problem to the development of Mori’s program. Hironaka [21] used the cone of effective curves to study the properties of birational morphisms. This direction was further developed and given a decisive impact by Mori [46], who introduced the notion of extremal rays and systematically used it in an attempt to construct minimal models in higher dimension, called *the minimal model program*. Danilov [16] introduced the notion of *canonical and terminal singularities* in conjunction with the toric factorization problem. This was developed by Reid into a general theory of these singularities [54, 55], which appear in an essential way in the minimal model program. The minimal model program is so far proven up to dimension 3 ([47], see also [31, 32, 33, 36, 62]), and for toric varieties in arbitrary dimension (See [56]). In the steps of the minimal model program one is only allowed to contract a divisor into a variety with terminal singularities, or to perform a flip, modifying some codimension  $\geq 2$  loci. This allows a factorization of a given birational morphism into such “elementary operations”. An algorithm to factor birational *maps* among uniruled varieties, known as *Sarkisov’s program*, has been developed and carried out in dimension 3 (see [59, 57, 10], and see [39] for the toric case). Still, we do not know of a way to solve the classical factorization problem using such a factorization.

**0.11. Relation with the toroidalization problem.** In [2], Theorem 2.1, it is proven that given a morphism of projective varieties  $X \rightarrow B$ , there are modifications  $m_X : X' \rightarrow X$  and  $m_B : B' \rightarrow B$ , with a lifting  $X' \rightarrow B'$  which has a toroidal structure. The *toroidalization problem* (see [2], [3], [35]) is that of obtaining such  $m_X$  and  $m_B$  which are composites of blowings up with smooth centers (maybe even with centers supported only over the locus where  $X \rightarrow B$  is not toroidal).

The proof in [2] relies on the work of De Jong [28] and methods of [1]. The authors of the present paper have tried to use these methods to approach the factorization conjectures, so far without success; one notion we do use in this paper is the toric ideal of [1]. It would be interesting if one could turn this approach on its head and prove a result on toroidalization using factorization.

**0.12. Relation with the proof in [70].** Another proof of the weak factorization theorem was given independently by the fourth author in [70]. The main difference of the two approaches is that in the current paper we are using objects such as toric ideals defined *locally* on each quasi-elementary piece of a cobordism. The blowing up of a toric ideal gives the quasi-elementary cobordism a toroidal structure. These toroidal modifications are then pieced together using canonical resolution of singularities. In [70] one works *globally*: a combinatorial description of *stratified toroidal varieties* and appropriate morphisms between them is given, which allows one to apply Morelli’s  $\pi$ -desingularization algorithm directly to the entire birational cobordism.

The structure of stratified toroidal variety on the cobordism is somewhere in between our notions of locally toric and toroidal structures.

**0.13. Outline of the paper.** In section 1 we discuss locally toric and toroidal structures, and reduce the proof of Theorem 0.1.1 to the case where  $\phi$  is a projective birational morphism.

Suppose now we have a projective birational morphism  $\phi : X_1 \rightarrow X_2$ . In section 2 we apply the theory of birational cobordisms to obtain a factorization into locally toric birational maps. Our cobordism  $B$  is relatively projective over  $X_2$ , and using a geometric invariant theory analysis, inspired by Thaddeus's work, we show that the intermediate varieties can be chosen to be projective over  $X_2$ .

In section 3 we utilize a factorization of the cobordism  $B$  into quasi-elementary pieces  $B_{a_i}$ , and for each piece construct an ideal sheaf  $I$  whose blowing up *torifies* the action of  $K^*$  on  $B_{a_i}$ . In other words,  $K^*$  acts toroidally on the variety obtained by blowing up  $B_{a_i}$  along  $I$ .

In section 4 we prove the weak factorization theorem by putting together the toroidal birational transforms induced by the quasi-elementary cobordisms. This is done using canonical resolution of singularities.

In section 5 we prove Theorem 0.3.1. We then discuss some problems related to strong factorization in section 6.

## 1. PRELIMINARIES

**1.1. Quotients.** Suppose a reductive group  $G$  acts on an algebraic variety  $X$ . We denote by  $X/G$  the space of orbits, and by  $X//G$  the space of equivalence classes of orbits, where the equivalence relation is generated by the condition that two orbits are equivalent if their closures intersect; such a space is endowed with a scheme structure which satisfies the usual universal property, if such a structure exists.

In this paper we will only consider  $X//G$  in situations where the closure of any orbit contains a unique closed orbit (see Definition 2.1.4). Moreover, the quotient morphism  $X \rightarrow X//G$  will be affine. When this holds we say that the action of  $G$  on  $X$  is *relatively affine*.

**1.2. Canonical resolution of singularities and canonical principalization.** In the following, we will use canonical versions of Hironaka's theorems on resolution of singularities and principalization of an ideal, proved in [7, 67].

**1.2.1. Canonical resolution.** A canonical embedded resolution of singularities  $\widetilde{W} \rightarrow W$  is a desingularization procedure consisting of a composite of blowings up with smooth centers, satisfying a number of conditions. In particular

1. "embedded" means the following: assume the sequence of blowings up is applied when  $W \subset U$  is a closed embedding with  $U$  nonsingular. Denote by  $E_i$  the exceptional divisor at some stage of the blowing up. Then **(a)**  $E_i$  is a normal crossings divisor, and has normal crossings with the center of blowing up, and **(b)** at the last stage  $\widetilde{W}$  has normal crossings with  $E_i$ .
2. "Canonical" means "functorial with respect to smooth maps", namely, if  $\theta : V \rightarrow W$  is a smooth morphism then the ideals blown up are invariant under pulling back by  $\theta$ ; hence  $\theta$  can be lifted to a smooth morphism  $\widetilde{\theta} : \widetilde{V} \rightarrow \widetilde{W}$ .

In particular: **(a)** if  $\theta : W \rightarrow W$  is an automorphism (of schemes, not necessarily over  $K$ ) then it can be lifted to an automorphism  $\widetilde{W} \rightarrow \widetilde{W}$ , and **(b)** the canonical resolution behaves well with respect to étale morphisms: if  $V \rightarrow W$  is étale, we get an étale morphism of canonical resolutions  $\widetilde{V} \rightarrow \widetilde{W}$ .

An important consequence of these conditions is that *all the centers of blowing up lie over the singular locus of  $W$* .

**1.2.2. Compatibility with a normal crossings divisor.** If  $W \subset U$  is embedded in a nonsingular variety, and  $D \subset U$  is a normal crossings divisor, then a variant of the resolution procedure allows one to choose the centers of blowing up to have normal crossings with  $D_i + E_i$ , where  $D_i$  is the inverse image of  $D$ . This follows since the resolution setup, as in [7], allows including such a divisor in "year 0".

**1.2.3. Principalization.** By *canonical principalization of an ideal sheaf* in a nonsingular variety we mean “the canonical embedded resolution of singularities of the subscheme defined by the ideal sheaf making it a divisor with normal crossings”; i.e., a composite of blowings up with smooth centers such that the total transform of the ideal is a divisor with simple normal crossings. Canonical embedded resolution of singularities of an arbitrary subscheme, not necessarily reduced or irreducible, is discussed in Section 11 of [7], and this implies canonical principalization, as one simply needs to blow up  $\widetilde{W}$  at the last step.

**1.2.4. Elimination of indeterminacies.** Now let  $\phi : W_1 \dashrightarrow W_2$  be a proper birational map between nonsingular varieties, and  $U \subset W_1$  an open set on which  $\phi$  restricts to an isomorphism. By *elimination of indeterminacies* of  $\phi$  we mean a morphism  $e : W'_1 \rightarrow W_1$ , obtained by a sequence of blowings up with smooth centers disjoint from  $U$ , such that the birational map  $\phi \circ e$  is a morphism.

Elimination of indeterminacies can be reduced to principalization of an ideal sheaf as follows.

We may assume that  $\phi^{-1}$  is a morphism; otherwise we replace  $W_2$  by the closure of the graph of  $\phi$ . Now we use Chow’s lemma (Corollary 2, p. 504, [24]): there exists an ideal sheaf  $I$  on  $W_1$  such that the blowing up of  $W_1$  along  $I$  factors through  $W_2$ . Hence the canonical principalization of  $I$  also factors through  $W_2$ .

Although it is not explicitly stated by Hironaka, the ideal  $I$  is supported in the complement of the open set  $U$ : the blowing up of  $I$  consists of a sequence of *permissible* blowings up (Definition 4.4.3, p. 537, [24]), each of which is supported in the complement of  $U$ . Another important fact is, that the ideal  $I$  is *invariant*, namely, it is functorial under absolute isomorphisms: if  $\phi : W'_1 \dashrightarrow W'_2$  is another proper birational map, with corresponding ideal  $I'$ , and  $\theta_i : W_i \rightarrow W'_i$  are isomorphisms such that  $\phi' \circ \theta_1 = \theta_2 \circ \phi$ , then  $\theta_1^* I' = I$ . This follows simply because at no point in Hironaka’s flattening procedure there is a need for any choice.

The same results hold for analytic and algebraic spaces. While Hironaka states his result only in the analytic setting, the arguments hold in the algebraic setting as well. See [53] for an earlier treatment of the case of varieties.

**1.3. Reduction to projective morphisms.** We start with a birational map

$$\phi : X_1 \dashrightarrow X_2$$

between complete nonsingular algebraic varieties  $X_1$  and  $X_2$  defined over  $K$  and restricting to an isomorphism on an open set  $U$ .

**Lemma 1.3.1** (Hironaka). *There is a commutative diagram*

$$\begin{array}{ccc} X'_1 & \xrightarrow{\phi'} & X'_2 \\ g_1 \downarrow & & \downarrow g_2 \\ X_1 & \dashrightarrow^{\phi} & X_2 \end{array}$$

such that  $g_1$  and  $g_2$  are composites of blowings up with smooth centers disjoint from  $U$ , and  $\phi'$  is a projective birational morphism.

**Proof.** By Hironaka’s theorem on elimination of indeterminacies, there is a morphism  $g_2 : X'_2 \rightarrow X_2$  which is a composite of blowings up with smooth centers disjoint from  $U$ , such that the birational map  $h := \phi^{-1} \circ g_2 : X'_2 \rightarrow X_1$  is a morphism:

$$\begin{array}{ccc} & X'_2 & \\ h \swarrow & \downarrow g_2 & \\ X_1 & \dashrightarrow^{\phi} & X_2 \end{array}$$

By the same theorem, there is a morphism  $g_1 : X'_1 \rightarrow X_1$  which is a composite of blowings up with smooth centers disjoint from  $U$ , such that  $\phi' := h^{-1} \circ g_1 : X'_1 \rightarrow X'_2$  is a morphism. Since the composite  $h \circ \phi' = g_1$  is projective, it follows that  $\phi'$  is projective.  $\square$

Thus we may replace  $X_1 \dashrightarrow X_2$  by  $X'_1 \rightarrow X'_2$  and assume from now on that  $\phi$  is a projective morphism.

Note that, by the properties of principalization and Hironaka’s flattening, the formation of  $\phi' : X'_1 \rightarrow X'_2$  is functorial under absolute isomorphisms, and the blowings up have normal crossings with the appropriate divisors. This will be used in the proof of Theorem 0.3.1.

**1.4. Toric varieties.** Let  $N \cong \mathbb{Z}^n$  be a lattice and  $\sigma \subset N_{\mathbb{R}}$  a strictly convex rational polyhedral cone. We denote the dual lattice by  $M$  and the dual cone by  $\sigma^{\vee} \subset M_{\mathbb{R}}$ . The *affine toric variety*  $X = X(N, \sigma)$  is defined as

$$X = \operatorname{Spec} K[M \cap \sigma^{\vee}].$$

For  $m \in M \cap \sigma^{\vee}$  we denote its image in the semigroup algebra  $K[M \cap \sigma^{\vee}]$  by  $z^m$ .

More generally, the toric variety corresponding to a fan  $\Sigma$  in  $N_{\mathbb{R}}$  is denoted by  $X(N, \Sigma)$ .

If  $X_1 = X(N, \Sigma_1)$  and  $X_2 = X(N, \Sigma_2)$  are two toric varieties, the embeddings of the torus  $T = \operatorname{Spec} K[M]$  in both of them define a toric (i.e.,  $T$ -equivariant) birational map  $X_1 \dashrightarrow X_2$ .

Suppose  $K^*$  acts effectively on an affine toric variety  $X = X(N, \sigma)$  as a one-parameter subgroup of the torus  $T$ , corresponding to a primitive lattice point  $a \in N$ . If  $t \in K^*$  and  $m \in M$ , the action on the monomial  $z^m$  is given by

$$t^*(z^m) = t^{(a, m)} \cdot z^m,$$

where  $(\cdot, \cdot)$  is the natural pairing on  $N \times M$ . The  $K^*$ -invariant monomials correspond to the lattice points  $M \cap a^{\perp}$ , hence

$$X//K^* \cong \operatorname{Spec} K[M \cap \sigma^{\vee} \cap a^{\perp}].$$

If  $a \notin \pm\sigma$  then  $\sigma^{\vee} \cap a^{\perp}$  is a full-dimensional cone in  $a^{\perp}$ , and it follows that  $X//K^*$  is again an affine toric variety, defined by the lattice  $\pi(N)$  and cone  $\pi(\sigma)$ , where  $\pi : N_{\mathbb{R}} \rightarrow N_{\mathbb{R}}/\mathbb{R} \cdot a$  is the projection. This quotient is a geometric quotient precisely when  $\pi : \sigma \rightarrow \pi(\sigma)$  is a bijection.

**1.5. Locally toric and toroidal structures.** There is some confusion in the literature between the notion of *toroidal embeddings* and toroidal morphisms ([34], [2]) and that of *toroidal varieties* (see [15]), which we prefer to call *locally toric varieties*, and locally toric morphisms between them. A crucial issue in this paper is the distinction between the two notions.

- Definition 1.5.1.** 1. A variety  $W$  is *locally toric* if for every closed point  $p \in W$  there exists an open neighborhood  $V_p \subset W$  of  $p$  and an étale morphism  $\eta_p : V_p \rightarrow X_p$  to a toric variety  $X_p$ . Such a morphism  $\eta_p$  is called a *toric chart* at  $p$ .
2. An open embedding  $U \subset W$  is a *toroidal embedding* if for every closed point  $p \in W$  there exists a toric chart  $\eta_p : V_p \rightarrow X_p$  at  $p$  such that  $U \cap V_p = \eta_p^{-1}(T)$ , where  $T \subset X_p$  is the torus. We call such charts *toroidal*. Sometimes we omit the open set  $U$  from the notation and simply say that a variety is toroidal.
3. We say that a locally toric (respectively, toroidal) chart on a variety is compatible with a divisor  $D \subset W$  if  $\eta_p^{-1}(T) \cap D = \emptyset$ , i.e.,  $D$  corresponds to a toric divisor on  $X_p$ .

- Definition 1.5.2.** 1. A proper birational morphism of locally toric varieties  $f : W_1 \rightarrow W_2$  is said to be *locally toric* if for every closed point  $q \in W_2$ , and any  $p \in f^{-1}q$ , there is a diagram of *fiber squares*

$$\begin{array}{ccccc} X_p & \leftarrow & V_p & \subset & W_1 \\ \phi \downarrow & & \downarrow & & \downarrow f \\ X_q & \leftarrow & V_q & \subset & W_2 \end{array}$$

where

- $\eta_p : V_p \rightarrow X_p$  is a toric chart at  $p$ ,
  - $\eta_q : V_q \rightarrow X_q$  is a toric chart at  $q$ , and
  - $\phi : X_p \rightarrow X_q$  is a toric morphism.
2. Let  $U_i \subset W_i$  ( $i = 1, 2$ ) be toroidal embeddings. A proper birational morphism  $f : W_1 \rightarrow W_2$  is said to be *toroidal*, if it satisfies the condition above for being locally toric, with toroidal charts. In particular,  $f^{-1}(U_2) = U_1$ .

**Remarks.** 1. A toroidal embedding as defined above is a toroidal embedding without self-intersection according to the definition in [34], and a birational toroidal morphism satisfies the condition of allowability in [34].

2. We note that this definition of a toroidal morphism, where the charts are chosen locally on the target  $W_2$ , *differs* from that in [2], where the charts are taken locally in the source  $W_1$ . It is a nontrivial fact, which we will not need in this paper, that *these notions do agree for proper birational morphisms*.

3. As the reader may notice, one can propose several variants of the definition above as well as the ones to come, and one can raise many subtle questions about comparison between the resulting notions. We will not address these issues in this paper at all.
4. To a toroidal embedding ( $U_W \subset W$ ) one can associate a polyhedral complex  $\Delta_W$ , such that proper birational toroidal morphisms to  $W$ , up to isomorphisms, are in one-to-one correspondence with certain subdivisions of the complex (see [34]). It follows from this that the composition of two proper birational toroidal morphisms  $W_1 \rightarrow W_2$  and  $W_2 \rightarrow W_3$  is again toroidal: the first morphism corresponds to a subdivision of  $\Delta_{W_2}$ , the second one to a subdivision of  $\Delta_{W_3}$ , hence their composition is the unique toroidal morphism corresponding to the subdivision  $\Delta_{W_1}$  of  $\Delta_{W_3}$ .
5. A composition of locally toric birational morphisms is not locally toric in general. A simple example is given by blowing up a point on a nonsingular threefold, and then blowing up a nonsingular curve tangent to the exceptional divisor.
6. One can define notions of locally toric and toroidal morphisms even when the morphisms are not proper or birational (see, e.g., [30], [2]). We will not need such notions in this paper.
7. Some of the issues we avoided discussing here are addressed in the third author's lecture notes [40].

**Definition 1.5.3** ([23],[27]). Let  $\psi : W_1 \dashrightarrow W_2$  be a rational map defined on a dense open subset  $U$ . Denote by  $\Gamma_\psi$  the closure of the graph of  $\psi_U$  in  $W_1 \times W_2$ . We say that  $\psi$  is *proper* if the projections  $\Gamma_\psi \rightarrow W_1$  and  $\Gamma_\psi \rightarrow W_2$  are both proper.

**Definition 1.5.4.** 1. A proper birational map  $\psi : W_1 \dashrightarrow W_2$  between two locally toric varieties  $W_1$  and  $W_2$  is said to be *locally toric* if there exists a locally toric variety  $Z$  and a commutative diagram

$$\begin{array}{ccc} & Z & \\ \swarrow & & \searrow \\ W_1 & \xrightarrow{\psi} & W_2 \end{array}$$

where  $Z \rightarrow W_i$  ( $i = 1, 2$ ) are proper birational locally toric morphisms.

2. Let  $U_i \subset W_i$  be toroidal embeddings. A proper birational map  $\psi : W_1 \dashrightarrow W_2$  is said to be *toroidal* if there exists a toroidal embedding  $U_Z \subset Z$  and a diagram as above where  $Z \rightarrow W_i$  are proper birational toroidal morphisms. In particular, a proper birational toroidal map induces an isomorphism between the open sets  $U_1$  and  $U_2$ .

*Remarks.* 1. It follows from the correspondence between toroidal modifications and subdivisions of polyhedral complexes that the composition of toroidal birational maps given by  $W_1 \leftarrow Z_1 \rightarrow W_2$  and  $W_2 \leftarrow Z_2 \rightarrow W_3$  is again toroidal. Indeed, if  $Z_1 \rightarrow W_2$  and  $Z_2 \rightarrow W_2$  correspond to two subdivisions of  $\Delta_{W_2}$ , then a common refinement of the two subdivisions corresponds to a toroidal embedding  $Z$  such that  $Z \rightarrow Z_1$  and  $Z \rightarrow Z_2$  are toroidal morphisms. For example, the coarsest refinement corresponds to taking for  $Z$  the normalization of the closure of the graph of the birational map  $Z_1 \dashrightarrow Z_2$ . The composite maps  $Z \rightarrow W_i$  are all toroidal birational morphisms.

2. It can be shown that a toroidal birational map which is regular is a toroidal morphism, therefore definitions 1.5.2 and 1.5.4 are compatible in the toroidal situation. We do not know if this is true for locally toric maps.

A composition of locally toric birational maps is not locally toric in general. Even worse, the locally toric structures on the morphisms  $Z \rightarrow W_i$  may be given with respect to incompatible toric charts in  $Z$ . Hence, for points  $p \in W_1$  and  $q \in W_2$  there may exist no toric charts at  $p$  and  $q$  in which the map  $\psi$  is given by a birational toric map.

To remedy this, we define a stronger version of locally toric and toroidal maps. These are the only maps we will need in the considerations of the current paper.

**Definition 1.5.5.** 1. A proper birational map  $\psi : W_1 \dashrightarrow W_2$  between locally toric varieties  $W_1$  and  $W_2$  is called *tightly locally toric* if there exists a locally toric variety  $Y$  and a commutative diagram

$$\begin{array}{ccc} W_1 & \xrightarrow{\psi} & W_2 \\ & \searrow & \swarrow \\ & Y & \end{array},$$

where  $W_i \rightarrow Y$  are proper birational locally toric morphisms, and for every closed point  $q \in Y$  there exists a toric chart  $\eta_q : V_q \rightarrow X_q$  at  $q$  such that the morphisms  $W_i \rightarrow Y$  can be given locally toric structures with respect to the same chart  $\eta_q$ .



2. Let  $U_i \subset W_i$  be toroidal embeddings. A proper birational map  $\psi : W_1 \dashrightarrow W_2$  is said to be *tightly toroidal* if there exists a toroidal embedding  $U_Y \subset Y$  and a diagram as above where  $W_i \rightarrow Y$  are proper birational toroidal morphisms.

*Remark.* The argument used before to show that a composition of toroidal birational maps is toroidal, shows that a tightly toroidal map is toroidal. A composition of tightly toroidal maps is not tightly toroidal in general. As for tightly locally toric maps, all varieties and morphisms can be given toroidal structures locally in  $Y$ . Now letting  $Z$  be the normalization of the closure of the graph of  $\psi$ , it follows that  $\psi$  is locally toric.

**1.6. Weak factorization for toroidal birational maps.** The weak factorization theorem for proper birational toric maps can be extended to the case of proper birational toroidal maps. This is proved in [3] for toroidal morphisms, using the correspondence between birational toroidal morphisms and subdivisions of polyhedral complexes. The general case of a toroidal birational map  $W_1 \leftarrow Z \rightarrow W_2$  can be deduced from this, as follows. By toroidal resolution of singularities we may assume  $Z$  is nonsingular. We apply toroidal weak factorization to the morphisms  $Z \rightarrow W_i$ , to get a sequence of toroidal birational maps

$$W_1 = V_1 \dashrightarrow V_2 \dashrightarrow \cdots \dashrightarrow V_{l-1} \dashrightarrow V_l = Z \dashrightarrow V_{l+1} \dashrightarrow \cdots \dashrightarrow V_{k-1} \dashrightarrow V_k = W_2$$

consisting of smooth toroidal blowings up and down.

We state this result for later reference:

**Theorem 1.6.1.** *Let  $U_1 \subset W_1$  and  $U_2 \subset W_2$  be nonsingular toroidal embeddings. Let  $\psi : W_1 \dashrightarrow W_2$  be a proper toroidal birational map. Then  $\psi$  can be factored into a sequence of toroidal birational maps consisting of smooth toroidal blowings up and down.*

This does not immediately imply that one can choose a factorization satisfying a projectivity statement as in the main theorem, or in a functorial manner. We will show these facts in Sections 2.7 and 5, respectively. It should be mentioned that if toric *strong* factorization is true, then the toroidal case follows.

### 1.7. Locally toric and toroidal actions.

**Definition 1.7.1** (see [48], p. 198). Let  $V$  and  $X$  be affine varieties with  $K^*$ -actions, and let  $\eta : V \rightarrow X$  be a  $K^*$ -equivariant étale morphism. Then  $\eta$  is said to be *strongly étale* if

- (i) the quotient map  $V//K^* \rightarrow X//K^*$  is étale, and
- (ii) the natural map

$$V \rightarrow X \times_{X//K^*} V//K^*$$

is an isomorphism.

**Definition 1.7.2.** 1. Let  $W$  be a locally toric variety with a  $K^*$ -action, such that  $W//K^*$  exists. We say that the action is *locally toric* if for any closed point  $p \in W$  we have a toric chart  $\eta_p : V_p \rightarrow X_p$  at  $p$ , and a one-parameter subgroup  $K^* \subset T_p$  of the torus in  $X_p$ , satisfying

- $V_p = \pi^{-1}\pi V_p$ , where  $\pi : W \rightarrow W//K^*$  is the projection;
- $\eta_p$  is  $K^*$ -equivariant and strongly étale.

2. If  $U \subset W$  is a toroidal embedding, we say that  $K^*$  acts toroidally on  $W$  if the charts above can be chosen toroidal.

The definition above is equivalent to the existence of the following diagram of fiber squares:

$$\begin{array}{ccccc} X_p & \leftarrow & V_p & \subset & W \\ \downarrow & & \downarrow & & \downarrow f \\ X//K^* & \leftarrow & V_p//K^* & \subset & W//K^* \end{array}$$

where the horizontal maps provide toric (resp. toroidal) charts in  $W$  and  $W//K^*$ . It follows that the quotient of a locally toric variety by a locally toric action is again locally toric; the same holds in the toroidal case.

*Remark.* If we do not insist on the charts being strongly étale, then the morphism of quotients may fail to be étale. Consider, for instance, the space  $X = \text{Spec } K[x, x^{-1}, y]$  with the action  $t(x, y) = (t^2x, t^{-1}y)$ . The quotient is  $X/K^* = \text{Spec } K[xy^2]$ . There is an equivariant étale cover  $V = \text{Spec } K[u, u^{-1}, y]$  with the action  $t(u, y) = (tu, t^{-1}y)$ , where the map is defined by  $x = u^2$ . The quotient is  $V/K^* = \text{Spec } K[uy]$ , which is a branched cover of  $X/K^*$ , since  $xy^2 = (uy)^2$ .

The following lemma shows that locally toric  $K^*$ -actions are ubiquitous. We note that it can be proven with fewer assumptions, see [69], [40].

**Lemma 1.7.3.** *Let  $W$  be a nonsingular variety with a relatively affine  $K^*$ -action, that is, the scheme  $W//K^*$  exists and the morphism  $W \rightarrow W//K^*$  is an affine morphism. Then the action of  $K^*$  on  $W$  is locally toric.*

**Proof.** Taking an affine open in  $W//K^*$ , we may assume that  $W$  is affine. We embed  $W$  equivariantly into a projective space and take its completion (see, e.g., [63]). After applying equivariant resolution of singularities to this completion (see Section 1.2) we may also assume that  $\overline{W}$  is a nonsingular projective variety with a  $K^*$ -action, and  $W \subset \overline{W}$  is an affine invariant open subset.

Let  $p \in W$  be a closed point. Since  $\overline{W}$  is complete, the orbit of  $p$  has a limit point  $q = \lim_{t \rightarrow 0} t(p)$  in  $\overline{W}$ . Now  $q$  is fixed by  $K^*$ , hence  $K^*$  acts on the cotangent space  $m_q/m_q^2$  at  $q$ . Since  $K^*$  is reductive, we can lift a set of eigenvectors of this action to semi-invariant local parameters  $x_1, \dots, x_n$  at  $q$ . These local parameters define a  $K^*$ -equivariant étale morphism  $\eta_q : V_q \rightarrow X_q$  from an affine  $K^*$  invariant open neighborhood  $V_q$  of  $q$  to the tangent space  $X_q = \text{Spec}(\text{Sym } m_q/m_q^2)$  at  $q$ . The latter has a structure of a toric variety, where the torus is the complement of the zero set of  $\prod x_i$ .

Separating the parameters  $x_i$  into  $K^*$ -invariants and non-invariants, we get a factorization  $X_q = X_q^0 \times X_q^1$ , where the action of  $K^*$  on  $X_q^1$  is trivial, and the action on  $X_q^0$  has 0 as its unique fixed point. Thus we get a product decomposition  $X_q//K^* = X_q^0//K^* \times X_q^1$ .

By Luna's Fundamental Lemma ([38], Lemme 3), there exist affine  $K^*$ -invariant neighborhoods  $V'_q$  of  $q$  and  $X'_q$  of 0, such that the restriction  $\eta'_q : V'_q \rightarrow X'_q$  is strongly étale. Consider first the case  $q \in W$ , in which case we may replace  $p$  by  $q$ . Denote  $Z = X_q^{K^*} \cap X'_q$ . Then  $Z \subset X_q^{K^*} \simeq X_q^1$  is affine open, and, using the direct product decomposition above,  $X_q^0 \times Z \subset X_q$  is affine open. Denote  $X''_q = X'_q \cap X_q^0 \times Z$ . This is affine open in  $X_q$ , and it is easy to see that  $X''_q//K^* \rightarrow X_q//K^*$  is an open embedding: an orbit in  $X''_q$  is closed if and only if it is closed in  $X_q$ . Writing  $V''_q = \eta'^{-1}_q X''_q$ , it follows that  $V''_q \rightarrow X_q$  is a strongly étale toric chart.

In case  $q \notin W$ , replace  $V_q$  by  $V''_q$ . Now  $\eta_q$  is injective on any orbit, and therefore it is injective on the orbit of  $p$ . Let  $X_p \subset X_q$  be the affine open toric subvariety in which the torus orbit of  $\eta_q(p)$  is closed, and let  $V_p = \eta_q^{-1} X_p \cap W$ . Now consider the restriction  $\eta : V_p \rightarrow X_p$ , where the  $K^*$ -orbits of  $p$  and  $\eta(p)$  are closed. By Luna's Fundamental Lemma there exist affine open  $K^*$ -invariant neighborhoods  $V'_p \subset V_p$  and  $X'_p \subset X_p$  of  $\eta_p(p)$  such that the restriction  $\eta : V'_p \rightarrow X'_p$  is a strongly étale morphism. Since  $X_p/K^*$  is a geometric quotient, we may assume  $X'_p = X_p$  and we have a strongly étale toric chart.

It remains to show that the charts can be chosen saturated with respect to the projection  $\pi : W \rightarrow W//K^*$ . If the orbit of  $p$  has a limit point  $q = \lim_{t \rightarrow 0} t \cdot p$  or  $q = \lim_{t \rightarrow \infty} t \cdot p$  in  $W$ , which is necessarily unique as  $\pi$  is affine, then an equivariant toric chart at  $q$  also covers  $p$ . So we may replace  $p$  by  $q$  and assume that the orbit of  $p$  is closed. Now  $\pi(W \setminus V_p)$  is closed and does not contain  $\pi(p)$ , so we can choose an affine neighborhood  $Y$  in its complement, and replace  $V_p$  by  $\pi^{-1}Y$ .  $\square$

## 2. BIRATIONAL COBORDISMS

### 2.1. Definitions.

**Definition 2.1.1** ([69]). Let  $\phi : X_1 \dashrightarrow X_2$  be a birational map between two algebraic varieties  $X_1$  and  $X_2$  over  $K$ , isomorphic on an open set  $U$ . A normal algebraic variety  $B$  is called a *birational cobordism* for  $\phi$  and denoted by  $B_\phi(X_1, X_2)$  if it satisfies the following conditions.

1. The multiplicative group  $K^*$  acts effectively on  $B = B_\phi(X_1, X_2)$ .
2. The sets

$$\begin{aligned} B_- &:= \{x \in B : \lim_{t \rightarrow 0} t(x) \text{ does not exist in } B\} \\ \text{and } B_+ &:= \{x \in B : \lim_{t \rightarrow \infty} t(x) \text{ does not exist in } B\} \end{aligned}$$

are nonempty Zariski open subsets of  $B$ .

3. There are isomorphisms

$$B_-/K^* \xrightarrow{\sim} X_1 \quad \text{and} \quad B_+/K^* \xrightarrow{\sim} X_2.$$

4. Considering the rational map  $\psi : B_- \dashrightarrow B_+$  induced by the inclusions  $B_- \cap B_+ \subset B_-$  and  $B_- \cap B_+ \subset B_+$ , the following diagram commutes:

$$\begin{array}{ccc} B_- & \xrightarrow{\psi} & B_+ \\ \downarrow & & \downarrow \\ X_1 & \xrightarrow{\phi} & X_2 \end{array}.$$

We say that  $B$  respects the open set  $U$  if  $U$  is contained in the image of  $(B_- \cap B_+)/K^*$ .

**Definition 2.1.2** ([69]). Let  $B = B_\phi(X_1, X_2)$  be a birational cobordism, and let  $F \subset B^{K^*}$  be a subset of the fixed-point set. We define

$$\begin{aligned} F^+ &= \{x \in B \mid \lim_{t \rightarrow 0} t(x) \in F\} \\ F^- &= \{x \in B \mid \lim_{t \rightarrow \infty} t(x) \in F\} \\ F^\pm &= F^+ \cup F^- \\ F^* &= F^\pm \setminus F \end{aligned}$$

**Definition 2.1.3** ([69]). Let  $B = B_\phi(X_1, X_2)$  be a birational cobordism. We define a relation  $\prec$  among connected components of  $B^{K^*}$  as follows: let  $F_1, F_2 \subset B^{K^*}$  be two connected components, and set  $F_1 \prec F_2$  if there is a point  $x \notin B^{K^*}$  such that  $\lim_{t \rightarrow 0} t(x) \in F_1$  and  $\lim_{t \rightarrow \infty} t(x) \in F_2$ .

**Definition 2.1.4.** A birational cobordism  $B = B_\phi(X_1, X_2)$  is said to be *quasi-elementary* if any two connected components  $F_1, F_2 \subset B^{K^*}$  are incomparable with respect to  $\prec$ .

Note that this condition prohibits, in particular, the existence of a “loop”, namely a connected component  $F$  and a point  $y \notin F$  such that both  $\lim_{t \rightarrow 0} t(x) \in F$  and  $\lim_{t \rightarrow \infty} t(x) \in F$ .

**Definition 2.1.5** ([69]). A quasi-elementary cobordism  $B$  is said to be *elementary* if the fixed point set  $B^{K^*}$  is connected.

**Definition 2.1.6** (cf. [44],[69]). We say that a birational cobordism  $B = B_\phi(X_1, X_2)$  is *collapsible* if the relation  $\prec$  is a strict pre-order, namely, there is no cyclic chain of fixed point components

$$F_1 \prec F_2 \prec \dots \prec F_m \prec F_1.$$

**2.2. The main example.** We now recall a fundamental example of an elementary birational cobordism in the toric setting, discussed in [69]:

**Example 2.2.1.** Let  $B = \mathbb{A}^n = \text{Spec } K[z_1, \dots, z_n]$  and let  $t \in K^*$  act by

$$t(z_1, \dots, z_i, \dots, z_n) = (t^{\alpha_1} z_1, \dots, t^{\alpha_i} z_i, \dots, t^{\alpha_n} z_n).$$

We assume  $K^*$  acts effectively, namely  $\gcd(\alpha_1, \dots, \alpha_n) = 1$ . We regard  $\mathbb{A}^n$  as a toric variety defined by a lattice  $N \cong \mathbb{Z}^n$  and a regular cone  $\sigma \in N_{\mathbb{R}}$  generated by the standard basis

$$\sigma = \langle v_1, \dots, v_n \rangle.$$

The dual cone  $\sigma^\vee$  is generated by the dual basis  $v_1^*, \dots, v_n^*$ , and we identify  $z^{v_i^*} = z_i$ . The  $K^*$ -action then corresponds to a one-parameter subgroup

$$a = (\alpha_1, \dots, \alpha_n) \in N.$$

We assume that  $a \notin \pm\sigma$ . We have the obvious description of the sets  $B_+$  and  $B_-$ :

$$\begin{aligned} B_- &= \{(z_1, \dots, z_n); z_i \neq 0 \text{ for some } i \text{ with } \alpha_i = (v_i^*, a) < 0\}, \\ B_+ &= \{(z_1, \dots, z_n); z_i \neq 0 \text{ for some } i \text{ with } \alpha_i = (v_i^*, a) > 0\}. \end{aligned}$$

We define the upper boundary and lower boundary fans of  $\sigma$  to be

$$\begin{aligned} \partial_- \sigma &= \{x \in \sigma; x + \epsilon \cdot a \notin \sigma \text{ for all } \epsilon > 0\}, \\ \partial_+ \sigma &= \{x \in \sigma; x + \epsilon \cdot (-a) \notin \sigma \text{ for all } \epsilon > 0\}. \end{aligned}$$

Then we obtain the description of  $B_+$  and  $B_-$  as the toric varieties corresponding to the fans  $\partial_+ \sigma$  and  $\partial_- \sigma$  in  $N_{\mathbb{R}}$ .

Let  $\pi : N_{\mathbb{R}} \rightarrow N_{\mathbb{R}}/\mathbb{R} \cdot a$  be the projection. Then  $B//K^*$  is again an affine toric variety defined by the lattice  $\pi(N)$  and cone  $\pi(\sigma)$ . Similarly, one can check that the geometric quotients  $B_-/K^*$  and  $B_+/K^*$  are

toric varieties defined by fans  $\pi(\partial_+\sigma)$  and  $\pi(\partial_-\sigma)$ . Since both  $\pi(\partial_+\sigma)$  and  $\pi(\partial_-\sigma)$  are subdivisions of  $\pi(\sigma)$ , we get a diagram of birational toric maps

$$\begin{array}{ccc} B_-/K^* & \xrightarrow{\varphi} & B_+/K^* \\ & \searrow \quad \swarrow & \\ & B//K^* & \end{array}$$

More generally, one can prove that if  $\Sigma$  is a subdivision of a convex polyhedral cone in  $N_{\mathbb{R}}$  with lower boundary  $\partial_-\Sigma$  and upper boundary  $\partial_+\Sigma$  relative to an element  $a \in N \searrow \pm \Sigma$ , then the toric variety corresponding to  $\Sigma$ , with the  $K^*$ -action given by the one-parameter subgroup  $a \in N$ , is a birational cobordism between the two toric varieties corresponding to  $\pi(\partial_-\Sigma)$  and  $\pi(\partial_+\Sigma)$  as fans in  $N_{\mathbb{R}}/\mathbb{R} \cdot a$ .

For the details, we refer the reader to [44], [69] and [3].  $\square$

**2.3. Construction of a cobordism.** It was shown in [69] that birational cobordisms exist for any birational map  $X_1 \dashrightarrow X_2$ . Here we deal with a very special case.

**Theorem 2.3.1.** *Let  $\phi : X_1 \rightarrow X_2$  be a projective birational morphism between complete nonsingular algebraic varieties, which is an isomorphism on an open set  $U$ . Then there is a complete nonsingular algebraic variety  $\overline{B}$  with an effective  $K^*$ -action, satisfying the following properties:*

1. *There exist closed embeddings  $\iota_1 : X_1 \hookrightarrow \overline{B}^{K^*}$  and  $\iota_2 : X_2 \hookrightarrow \overline{B}^{K^*}$  with disjoint images.*
2. *The open subvariety  $B = \overline{B} \searrow (\iota_1(X_1) \cup \iota_2(X_2))$  is a birational cobordism between  $X_1$  and  $X_2$  respecting the open set  $U$ .*
3. *There is a coherent sheaf  $E$  on  $X_2$ , with a  $K^*$ -action, and a closed  $K^*$ -equivariant embedding  $\overline{B} \subset \mathbb{P}(E) := \text{Proj}_{X_2} \text{Sym} E$ .*

**Proof.** Let  $J \subset \mathcal{O}_{X_2}$  be an ideal sheaf such that  $\phi : X_1 \rightarrow X_2$  is the blowing up morphism of  $X_2$  along  $J$  and  $J_U = \mathcal{O}_U$ . Let  $I_0$  be the ideal of the point  $0 \in \mathbb{P}^1$ . Consider  $W_0 = X_2 \times \mathbb{P}^1$  and let  $p : W_0 \rightarrow X_2$  and  $q : W_0 \rightarrow \mathbb{P}^1$  be the projections. Let  $I = (p^{-1}J + q^{-1}I_0)\mathcal{O}_{W_0}$ . Let  $W$  be the blowing up of  $W_0$  along  $I$ . (Paolo Aluffi has pointed out that this  $W$  is used when constructing the deformation to the normal cone of  $J$ .)

We claim that  $X_1$  and  $X_2$  lie in the nonsingular locus of  $W$ . For  $X_2 \cong X_2 \times \{\infty\} \subset X_2 \times \mathbb{A}^1 \subset W$  this is clear. Since  $X_1$  is nonsingular, embedded in  $W$  as the strict transform of  $X_2 \times \{0\} \subset X_2 \times \mathbb{P}^1$ , to prove that  $X_1$  lies in the nonsingular locus, it suffices to prove that  $X_1$  is a Cartier divisor in  $W$ . We look at local coordinates. Let  $A = \Gamma(V, \mathcal{O}_V)$  for some affine open subset  $V \subset X_2$ , and let  $y_1, \dots, y_m$  be a set of generators of  $J$  on  $V$ . Then on the affine open subset  $V \times \mathbb{A}^1 \subset X_2 \times \mathbb{P}^1$  with coordinate ring  $A[x]$ , the ideal  $I$  is generated by  $y_1, \dots, y_m, x$ . The charts of the blowing up containing the strict transform of  $\{x = 0\}$  are of the form

$$\text{Spec } A \left[ \frac{y_1}{y_i}, \dots, \frac{y_m}{y_i}, \frac{x}{y_i} \right] = \text{Spec } A \left[ \frac{y_1}{y_i}, \dots, \frac{y_m}{y_i} \right] \times \text{Spec } K \left[ \frac{x}{y_i} \right],$$

where  $K^*$  acts on the second factor. The strict transform of  $\{x = 0\}$  is defined by  $\frac{x}{y_i}$ , hence it is Cartier.

Let  $\overline{B} \rightarrow W$  be a canonical resolution of singularities. Then conditions 1 and 2 are clearly satisfied. For condition 3, note that  $\overline{B} \rightarrow X_2 \times \mathbb{P}^1$ , being a composition of blowings up of invariant ideals, admits an equivariant ample line bundle. Twisting by the pullback of  $\mathcal{O}_{\mathbb{P}^1}(n)$  we obtain an equivariant line bundle which is ample for  $\overline{B} \rightarrow X_2$ . Replacing this by a sufficiently high power and pushing forward we get  $E$ .  $\square$

We refer the reader to [69] for more details.

We call a variety  $\overline{B}$  as in the theorem a *compactified, relatively projective cobordism*.

**2.4. Collapsibility and Projectivity.** Let  $B = B_{\phi}(X_1, X_2)$  be a birational cobordism. We seek a criterion for collapsibility of  $B$ .

Let  $\mathcal{C}$  be the set of connected components of  $B_{\phi}(X_1, X_2)^{K^*}$ , and let  $\chi : \mathcal{C} \rightarrow \mathbb{Z}$  be a function. We say that  $\chi$  is strictly increasing if  $F \prec F' \Rightarrow \chi(F) < \chi(F')$ . The following lemma is obvious:

**Lemma 2.4.1.** *Assume there exists a strictly increasing function  $\chi$ . Then  $\prec$  is a strict pre-order, and  $B$  is collapsible. Conversely, suppose  $B$  is collapsible. Then there exists a strictly increasing function  $\chi$ .*  $\square$

*Remark.* It is evident that every strictly increasing function can be replaced by one which induces a strict total order. However, it will be convenient for us to consider arbitrary strictly increasing functions.

Let  $\chi$  be a strictly increasing function, and let  $a_0 < a_1 < \dots < a_m \in \mathbb{Z}$  be the values of  $\chi$ .

**Definition 2.4.2.** We denote

1.  $F_{a_i} = \cup \{F \mid \chi(F) = a_i\}$ .
2.  $F_{a_i}^+ = \cup \{F^+ \mid \chi(F) = a_i\}$ .
3.  $F_{a_i}^- = \cup \{F^- \mid \chi(F) = a_i\}$ .
4.  $F_{a_i}^\pm = \cup \{F^\pm \mid \chi(F) = a_i\}$ .
5.  $F_{a_i}^* = \cup \{F^* \mid \chi(F) = a_i\}$ .
6.  $B_{a_i} = B \setminus (\cup \{F^- \mid \chi(F) < a_i\} \cup \cup \{F^+ \mid \chi(F) > a_i\})$ .

Note that  $F_{a_i}^*$  is the union of non-closed orbits in  $B_{a_i}$ . The following is an immediate extension of Proposition 1 of [69].

**Proposition 2.4.3.** 1.  $B_{a_i}$  is a quasi-elementary cobordism.  
 2. For  $i = 0, \dots, m-1$  we have  $(B_{a_i})_+ = (B_{a_{i+1}})_-$ .

The following is an analogue of Lemma 1 of [69] in the case of the cobordisms we have constructed.

**Proposition 2.4.4.** Let  $E$  be a coherent sheaf on  $X_2$  with a  $K^*$ -action, and let  $\overline{B} \subset \mathbb{P}(E)$  be a compactified, relatively projective cobordism embedded  $K^*$ -equivariantly. Then there exists a strictly increasing function  $\chi$  for the cobordism  $B = \overline{B} \setminus (X_1 \cup X_2)$ . In particular, the cobordism is collapsible.

**Proof.** Since  $K^*$  acts trivially on  $X_2$ , and since  $K^*$  is reductive, there exists a direct sum decomposition

$$E = \bigoplus_{b \in \mathbb{Z}} E_b$$

where  $E_b$  is the subsheaf on which the action of  $K^*$  is given by the character  $t \mapsto t^b$ . Denote by  $b_0, \dots, b_k$  the characters which figure in this representation. Note that there are disjoint embeddings  $\mathbb{P}(E_{b_j}) \subset \mathbb{P}(E)$ .

Let  $p \in B$  be a fixed point lying in the fiber  $\mathbb{P}(E_q)$  over  $q \in X_2$ . We choose a basis

$$(x_{b_0,1}, \dots, x_{b_0,d_0}, \dots, x_{b_k,1}, \dots, x_{b_k,d_k})$$

of  $E_q$  where  $x_{b_j,\nu} \in E_{b_j}$  and use the following lemma:

**Lemma 2.4.5.** Suppose  $p \in \mathbb{P}(E_q)^{K^*}$  is a fixed point with homogeneous coordinates

$$(p_{b_0,1}, \dots, p_{b_0,d_0}, \dots, p_{b_k,1}, \dots, p_{b_k,d_k}).$$

Then there is an  $j_p$  such that  $p_{b_j,\nu} = 0$  whenever  $j \neq j_p$ . In particular,  $p \in \mathbb{P}(E_{b_{j_p}}) \subset \mathbb{P}(E)$ . □

If  $F \subset B^{K^*}$  is a connected component of the fixed point set, then it follows from the lemma that  $F \subset \mathbb{P}(E_{b_j})$  for some  $j$ . We define

$$\chi(F) = b_j.$$

To check that  $\chi$  is strictly increasing, consider a point  $p \in B$  such that  $\lim_{t \rightarrow 0} t(p) \in F_1$  and  $\lim_{t \rightarrow \infty} t(p) \in F_2$  for some fixed point components  $F_1$  and  $F_2$ . Let the coordinates of  $p$  in the fiber over  $q \in X_2$  be  $(p_{b_0,1}, \dots, p_{b_0,d_0}, \dots, p_{b_k,1}, \dots, p_{b_k,d_k})$ . Now

$$\begin{aligned} \lim_{t \rightarrow 0} t(p) &\in \mathbb{P}(E_{b_{\min}}), \\ \lim_{t \rightarrow \infty} t(p) &\in \mathbb{P}(E_{b_{\max}}), \end{aligned}$$

where

$$\begin{aligned} b_{\min} &= \min\{b_j : p_{b_j,\nu} \neq 0 \text{ for some } \nu\}, \\ b_{\max} &= \max\{b_j : p_{b_j,\nu} \neq 0 \text{ for some } \nu\}. \end{aligned}$$

Thus, if  $p$  is not fixed by  $K^*$  then

$$\chi(F_1) = b_{\min} < b_{\max} = \chi(F_2).$$

□

**2.5. Geometric invariant theory and projectivity.** In this section we use geometric invariant theory, and ideas (originating in symplectic geometry) developed by M. Thaddeus and others (see e.g. [66]), in order to obtain a result about relative projectivity of quotients.

We continue with the notation of the last section. Consider the sheaf  $E$  and its decomposition according to the character. Let  $\{b_j\}$  be the characters of the action of  $K^*$  on  $E$ , and  $\{a_i\}$  the subset of those  $b_j$  that are in the image of  $\chi$ . If we use the Veronese embedding  $\overline{B} \subset \mathbb{P}(\text{Sym}^2(E))$  and replace  $E$  by  $\text{Sym}^2(E)$ , we may assume that  $a_i$  are even, in particular  $a_{i+1} > a_i + 1$  (this is a technical condition which comes up handy in what follows).

Denote by  $\rho_0(t)$  the action of  $t \in K^*$  on  $E$ . For any  $r \in \mathbb{Z}$  consider the “twisted” action  $\rho_r(t) = t^{-r} \cdot \rho_0(t)$ . Note that the induced action on  $\mathbb{P}(E)$  does not depend on the “twist”  $r$ . Considering the decomposition  $E = \bigoplus E_{b_j}$ , we see that  $\rho_r(t)$  acts on  $E_{b_j}$  by multiplication by  $t^{b_j - r}$ .

We can apply geometric invariant theory in its relative form (see, e.g., [51], [26]) to the action  $\rho_r(t)$  of  $K^*$ . Recall that a point  $p \in \mathbb{P}(E)$  is said to be semistable with respect to  $\rho_r$ , written  $p \in (\mathbb{P}(E), \rho_r)^{ss}$ , if there is a positive integer  $n$  and a  $\rho_r$ -invariant local section  $s \in (\text{Sym}^n(E))^{\rho_r}$ , such that  $s(p) \neq 0$ . The main result of geometric invariant theory implies that

$$\mathcal{P}roj_{X_2} \bigoplus_{n \geq 0}^{\infty} (\text{Sym}^n(E))^{\rho_r} = (\mathbb{P}(E), \rho_r)^{ss} // K^*;$$

moreover, the quotient map  $(\mathbb{P}(E), \rho_r)^{ss} \rightarrow (\mathbb{P}(E), \rho_r)^{ss} // K^*$  is affine. We can define  $(\overline{B}, \rho_r)^{ss}$  analogously, and we automatically have  $(\overline{B}, \rho_r)^{ss} = \overline{B} \cap (\mathbb{P}(E), \rho_r)^{ss}$ .

The numerical criterion of semistability (see [48]) immediately implies the following:

**Lemma 2.5.1.** *For  $0 < i < m$  we have*

1.  $(\overline{B}, \rho_{a_i})^{ss} = B_{a_i}$ ;
2.  $(\overline{B}, \rho_{a_i+1})^{ss} = (B_{a_i})_+$ ;
3.  $(\overline{B}, \rho_{a_i-1})^{ss} = (B_{a_i})_-$ ;

In other words, the triangle of birational maps

$$\begin{array}{ccc} (B_{a_i})_- / K^* & \xrightarrow{\varphi_i} & (B_{a_i})_+ / K^* \\ & \searrow \quad \swarrow & \\ & B_{a_i} // K^* & \end{array}$$

is induced by a change of linearization of the action of  $K^*$ .

In particular we obtain:

**Proposition 2.5.2.** *The morphisms  $(B_{a_i})_+ / K^* \rightarrow X_2$ ,  $(B_{a_i})_- / K^* \rightarrow X_2$  and  $B_{a_i} // K^* \rightarrow X_2$  are projective.*

**2.6. The main result of [69].** Let  $B$  be a collapsible nonsingular birational cobordism. Then we can write  $B$  as a union of quasi-elementary cobordisms  $B = \cup_i B_{a_i}$ , with  $(B_{a_i})_+ = (B_{a_{i+1}})_-$ . By Lemma 1.7.3 each  $B_{a_i}$  has a locally toric structure such that the action of  $K^*$  is locally toric.

**Lemma 2.6.1.** *Let  $B_{a_i}$  be a quasi-elementary cobordism, with a relatively affine locally toric  $K^*$  action. Then  $B_{a_i} // K^*$ ,  $(B_{a_i})_- / K^*$ ,  $(B_{a_i})_+ / K^*$  are locally toric varieties and we have a diagram of locally toric maps*

$$\begin{array}{ccc} (B_{a_i})_- / K^* & \xrightarrow{\varphi_i} & (B_{a_i})_+ / K^* \\ & \searrow \quad \swarrow & \\ & B_{a_i} // K^* & \end{array}$$

where  $\varphi_i$  is a tightly locally toric birational map.

In case  $B_{a_i}$  is nonsingular, the diagram above can be described in toric charts by the main example in Section 2.2.

If the action of  $K^*$  on  $B_{a_i}$  is toroidal then all these varieties and maps are also toroidal, and  $\varphi_i$  is a tightly toroidal birational map.

**Proof.** Let  $\eta_p : V_p \rightarrow X_p$  be a strongly étale  $K^*$ -equivariant toric chart in  $B_{a_i}$  giving a locally toric structure to the action of  $K^*$ . Then  $(V_p)_- = (B_{a_i})_- \cap V_p$  and the morphism  $(V_p)_- \rightarrow (X_p)_-$  is again strongly

étale, providing locally toric structures on the variety  $(B_{a_i})_-/K^*$  and the morphism  $(B_{a_i})_-/K^* \rightarrow B_{a_i}/K^*$ . Similarly for  $(B_{a_i})_+$ .  $\square$

Now we assume  $B \subset \overline{B}$  is open in a compactified, relatively projective cobordism. When we compose the birational transformations obtained from each  $B_{a_i}$  we get a slight refinement of the main result of [69].

**Theorem 2.6.2.** *Let  $\phi : X_1 \dashrightarrow X_2$  be a birational map between complete nonsingular algebraic varieties  $X_1$  and  $X_2$  over an algebraically closed field  $K$  of characteristic zero, and let  $U \subset X_1$  be an open set where  $\phi$  is an isomorphism. Then there exists a sequence of birational maps between complete locally toric algebraic varieties*

$$X_1 = W_0 \xrightarrow{\varphi_1} W_1 \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_i} W_i \xrightarrow{\varphi_{i+1}} W_{i+1} \xrightarrow{\varphi_{i+2}} \cdots \xrightarrow{\varphi_{m-1}} W_{m-1} \xrightarrow{\varphi_m} W_m = X_2$$

where

1.  $\phi = \varphi_m \circ \varphi_{m-1} \circ \cdots \circ \varphi_2 \circ \varphi_1$ ,
2.  $\varphi_i$  are isomorphisms on  $U$ , and
3. For each  $i$ , the map  $\varphi_i$  is tightly locally toric, and étale locally equivalent to a map  $\varphi$  described in 2.2.

Furthermore, there is an index  $i_0$  such that for all  $i \leq i_0$  the map  $W_i \dashrightarrow X_1$  is a projective morphism, and for all  $i \geq i_0$  the map  $W_i \dashrightarrow X_2$  is a projective morphism. In particular, if  $X_1$  and  $X_2$  are projective then all the  $W_i$  are projective.

*Remark.* For the projectivity claim 2, we take the first  $i_0$  terms in the factorization to come from Hironaka's elimination of indeterminacies in Lemma 1.3.1, which is projective over  $X_1$ , whereas the last terms come from  $\overline{B}$ , which is projective over  $X_2$ , and the geometric invariant theory considerations as in Proposition 2.5.2.

**2.7. Projectivity of toroidal weak factorization.** The following is a refinement of Theorem 1.6.1, in which a projectivity statement is added:

**Theorem 2.7.1.** *Let  $U_1 \subset W_1$  and  $U_2 \subset W_2$  be nonsingular toroidal embeddings. Let  $\psi : W_1 \dashrightarrow W_2$  be a proper toroidal birational map. Then  $\psi$  can be factored into a sequence of toroidal birational maps consisting of smooth toroidal blowings up and down, namely:*

$$W_1 = V_0 \xrightarrow{\varphi_1} V_1 \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_i} V_i \xrightarrow{\varphi_{i+1}} V_{i+1} \xrightarrow{\varphi_{i+2}} \cdots \xrightarrow{\varphi_{l-1}} V_{l-1} \xrightarrow{\varphi_l} V_l = W_2$$

where

1.  $\phi = \varphi_l \circ \varphi_{l-1} \circ \cdots \circ \varphi_2 \circ \varphi_1$ ;
2.  $\varphi_i$  are isomorphisms on  $U$ , the embeddings  $U \subset V_i$  are toroidal, and  $\varphi_i$  are toroidal birational maps; and
3. either  $\varphi_i : V_i \dashrightarrow V_{i+1}$  or  $\varphi_i^{-1} : V_{i+1} \dashrightarrow V_i$  is a toroidal morphism obtained by blowing up a smooth irreducible toroidal center.

Furthermore, there is an index  $i_0$  such that for all  $i \leq i_0$  the map  $V_i \dashrightarrow X_1$  is a projective morphism, and for all  $i \geq i_0$  the map  $V_i \dashrightarrow X_2$  is a projective morphism. In particular, if  $X_1$  and  $X_2$  are projective then all the  $V_i$  are projective.

**Proof.** As in [3], Lemma 8.7 we reduce to the case where the polyhedral complex of  $W_2$  is embeddable as a quasi-projective toric fan  $\Delta_2$  in a space  $N_{\mathbb{R}}$ . Indeed that Lemma gives an embedding preserving the  $\mathbb{Q}$ -structure for the barycentric subdivision of any simplicial complex, and since  $\Delta_2$  is nonsingular this embedding preserves integral structures as well. A further subdivision ensures that the fan is quasi-projective. (We note that this embedding is introduced for the sole purpose of applying Morelli's  $\pi$ -Desingularization Lemma directly, rather than observing that the proof works word for word in the toroidal case.)

As in 1.6.1 we may assume  $W_1 \dashrightarrow W_2$  is a projective morphism. Thus the complex  $\Delta_1$  of  $W_1$  is a projective subdivision of  $\Delta_2$ . Our construction of a compactified relatively projective cobordism  $\overline{B}$  for the morphism  $\phi$  yields a toroidal embedding  $B$  whose complex  $\Delta_B$  is a quasi-projective polyhedral cobordism lying in  $(N \oplus \mathbb{Z})_{\mathbb{R}}$  such that  $\pi(\partial_+ \Delta_B) = \Delta_2$  and  $\pi(\partial_- \Delta_B) = \Delta_1$ , where  $\pi$  is the projection onto  $N_{\mathbb{R}}$ . Moreover, the toroidal morphism  $B \rightarrow W_2$  gives a polyhedral morphism  $\Delta_B \rightarrow \Delta_2$  induced by the projection  $\pi$ . Morelli's  $\pi$ -desingularization lemma gives a projective subdivision  $\Delta'_B \rightarrow \Delta_B$ , isomorphic on the upper and lower boundaries  $\partial_{\pm} \Delta_B$ , such that  $\Delta'_B$  is  $\pi$ -nonsingular. We still have a polyhedral morphism  $\Delta'_B \rightarrow \Delta_2$ . The complex  $\Delta'_B$  corresponds to a toroidal birational cobordism  $B'$  between  $W_1$  and  $W_2$ . Since  $\Delta'_B$  is  $\pi$ -nonsingular, any elementary piece  $B'_F \subset B'$  corresponds a toroidal blowing up followed by a toroidal blowing down between nonsingular toroidal embeddings, with nonsingular centers. It follows that the same holds for

every quasi-elementary piece of  $B'$  (here the centers may be reducible). As in Theorem 2.6.2 above, these toroidal embeddings can be chosen to be projective over  $W_2$ .  $\square$

### 3. TORIFICATION

We wish to replace the locally toric factorization of Theorem 2.6.2 by a toroidal factorization. This amounts to replacing  $B$  with a locally toric  $K^*$ -action by some  $B'$  with a toroidal  $K^*$ -action. We call such a procedure *torification*. The basic idea, which goes back at least to Hironaka, is that if one blows up an ideal, the exceptional divisors provide the resulting variety with useful extra structure. The ideal we construct, called a *torific ideal*, is closely related to the torific ideal of [1].

#### 3.1. Construction of a torific ideal.

**Definition 3.1.1.** Let  $V$  be an algebraic variety with a  $K^*$ -action,  $p \in V$  a closed point,  $G_p \subset K^*$  the stabilizer of  $p$ . Fix an integer  $\alpha$ . Then we define

$$J_{\alpha,p} \subset \mathcal{O}_{V,p}$$

to be the ideal generated by the semi-invariant functions  $f \in \mathcal{O}_{V,p}$  of  $G_p$ -character  $\alpha$ , that is, for  $t \in G_p$  we have

$$t^*(f) = t^\alpha f.$$

**Lemma 3.1.2.** *Let  $V$  be a variety with a  $K^*$ -action, and  $p \in V$  a closed point. If  $z_1, \dots, z_n$  are  $G_p$ -semi-invariant generators of the maximal ideal  $\mathfrak{m}_p$ , then  $J_{\alpha,p}$  is generated by monomials in  $z_i$  having  $G_p$ -character  $\alpha$ .*

**Proof.** Consider the completion of the local ring  $\hat{\mathcal{O}}_{V,p}$ , and lift the action of  $G_p$  to it. Since  $\hat{\mathcal{O}}_{V,p}$  is a faithfully flat  $\mathcal{O}_{V,p}$ -module, it suffices to prove that the completion  $\hat{J}_{\alpha,p}$  is the ideal of  $\hat{\mathcal{O}}_{V,p}$  generated by monomials in  $z_i$  of  $G_p$ -character  $\alpha$ .

Consider the  $G_p$ -equivariant epimorphism

$$K[[z_1, \dots, z_n]] \rightarrow \hat{\mathcal{O}}_{V,p}.$$

Since  $G_p$  is reductive, a semi-invariant element of  $\hat{\mathcal{O}}_{V,p}$  is the image of a semi-invariant power series in  $z_i$ . A monomial in  $K[[z_1, \dots, z_n]]$  of  $G_p$ -character  $\alpha$  clearly maps to  $\hat{J}_{\alpha,p}$ . Conversely, a semi-invariant power series in  $z_i$  must have all its monomials semi-invariant of the same character. One can choose a finite set of monomials occurring in the power series such that any other monomial occurring in this power series is divisible by one of them. Hence the power series lies in the ideal generated by monomials in  $z_i$  of  $G_p$ -character  $\alpha$ .  $\square$

The lemma implies that, given a strongly étale morphism  $\eta : V \rightarrow X$  between varieties with  $K^*$  action, the inverse image of  $J_{\alpha,\eta(p)}$  generates  $J_{\alpha,p}$ . Indeed, we can choose  $G_p = G_{\eta(p)}$  semi-invariant generators of  $\mathfrak{m}_{\eta(p)}$ , which pull back to semi-invariant generators of  $\mathfrak{m}_p$ .

For the rest of this section, we let  $B$  be a *quasi-elementary* cobordism with a *relatively affine, locally toric*  $K^*$ -action;  $B = B_{a_i}$  for some  $i$  according to our previous notation. Without loss of generality we assume  $a_i = 0$ , so  $F_0 = B^{K^*}$ . Recall the notation  $F_0^*$  in Definition 2.4.2. This is a constructible set in  $B$ , which is the union of the non-closed orbits.

**Proposition 3.1.3.** *There exists a unique coherent  $K^*$ -equivariant ideal sheaf  $I_\alpha$  on  $B$ , such that for all  $p \in B \setminus F_0^*$  we have  $(I_\alpha)_p = J_{\alpha,p}$ .*

**Definition 3.1.4.** The sheaf  $I_\alpha$  is called the  $\alpha$ -torific ideal sheaf of the action of  $K^*$  on  $B$ .

*Remarks.* 1. Notice that the collection of ideals  $J_{\alpha,p}$  for  $p \in B$  does not define a coherent sheaf of ideals in general. As an example, let  $B = \mathbb{A}^2$ , and let  $t \in K^*$  act by

$$t(x, y) = (tx, t^{-1}y).$$

Then at  $p = (0, 0)$ , the stabilizer is  $G_p = K^*$ , and  $J_{1,p} = (x)$ . Any other point  $q \in B \setminus \{p\}$  has a trivial stabilizer, hence  $J_{1,q} = \mathcal{O}_{B,q}$  is trivial. These germs do not form a coherent ideal sheaf on  $B$ . In this case, the ideal sheaf generated by  $x$  is the 1-torific ideal sheaf  $I_1$  of the proposition.



2. Note also that the assertion of the proposition fails if we remove the requirement on  $B$  being quasi-elementary. For a simple example which is not a cobordism, let  $t \in K^*$  act on  $B = \mathbb{P}^1$  with homogeneous coordinates  $(X : Y)$ , via  $(X : Y) \rightarrow (tX : Y)$ . Then at  $p = (1 : 0)$  the ideal  $J_{1,p}$  is generated by  $x_0 = X/Y$ , whereas at  $q = (0 : 1)$  the ideal  $J_{1,q}$  is the zero ideal. It is easy to construct higher dimensional examples of cobordisms where the ideal  $J_{\alpha,p}$  at one fixed point cannot be glued to *any*  $\beta$ -toric ideal at another fixed point.

**Proof of 3.1.3.** First we prove the uniqueness of  $I_\alpha$ . Clearly the stalks of  $I_\alpha$  are uniquely defined at all points  $p \in B \setminus F_0^*$ . Now if  $p \in F_0^*$  then  $p$  has a unique limit fixed point  $p' \in F_0$ , where either  $p' = \lim_{t \rightarrow 0} t(p)$ , or  $p' = \lim_{t \rightarrow \infty} t(p)$ , but not both, since  $B$  is quasi-elementary. Since  $I_\alpha$  is uniquely determined at  $p'$ , hence also near  $p'$  by coherence, it follows from  $K^*$ -equivariance that  $I_\alpha$  is uniquely determined at  $p$ .

To prove the existence of  $I_\alpha$  we cover  $B$  with strongly étale affine toric charts  $\eta_p : V_p \rightarrow X_p$  as in Lemma 1.7.3. With such charts, it follows that  $F_0^*$  restricted to  $V_p$  is the inverse image of  $F_0^*$  defined in  $X_p$  (recall that  $F_0^*$  consists of the union of non-closed orbits). Lemma 3.1.5 below gives the existence of a toric ideal on  $X_p$ , and its pullback is a toric ideal on  $V_p$  by Lemma 3.1.2. By uniqueness, the ideals defined on  $V_p$  glue together to an ideal on  $B$ .  $\square$

**Lemma 3.1.5.** *Let  $B = X(N, \sigma)$  be an affine toric variety on which  $K^*$  acts as a one-parameter subgroup of the torus. Then  $I_\alpha$  exists and is generated by all monomials  $z^m, m \in \sigma^\vee$  on which  $K^*$  acts by character  $\alpha$ .*

**Proof.** By abuse of notation we will use the same letter  $\alpha$  to denote a character of a subgroup of  $K^*$ . It follows easily from Lemma 3.1.2 that for any  $p \in B$  the ideal  $J_{\alpha,p}$  is generated by all elements  $z^m$  regular at  $p$  on which  $G_p$  acts by character  $\alpha$ .

Let  $p \in B \setminus F_0^*$ , and let  $\tau$  be the smallest face of  $\sigma$  such that  $p$  lies in the affine open toric subvariety  $X(N, \tau)$ . Then the monomials  $z^m$  regular at  $p$  are those for which  $m \in M \cap \tau^\vee$ , and the monomials invertible at  $p$  are the ones for which  $m \in M \cap \tau^\perp$ .

If  $z^m$  for  $m \in M \cap \sigma^\vee$  is a monomial regular on  $B$  on which  $K^*$  acts by character  $\alpha$  then clearly  $z^m$  is regular at  $p$  and  $G_p \subset K^*$  acts on it by character  $\alpha$ . Conversely, let  $z^m$  for  $m \in M \cap \tau^\vee$  be a monomial regular at  $p$ , on which  $G_p$  acts by character  $\alpha$ . We show that there exists a monomial  $z^{m'}$  invertible at  $p$  (i.e.,  $m' \in M \cap \tau^\perp$ ) such that  $z^{m+m'}$  is regular on  $B$  and  $K^*$  acts on it by character  $\alpha$ . This is done in two steps:

STEP 1. There exists  $m' \in M \cap \tau^\perp$  such that  $z^{m+m'}$  has  $K^*$ -character  $\alpha$ . Since  $G_p$  is the subgroup of  $K^*$  acting trivially on the monomials corresponding to  $m' \in M \cap \tau^\perp$ , we have an exact sequence

$$M \cap \tau^\perp \rightarrow \hat{K}^* \rightarrow \hat{G}_p \rightarrow 0,$$

where  $\hat{H}$  denotes the character group of  $H$ . Thus, we may replace  $m$  by  $m + m'$  and assume that  $K^*$  acts on  $z^m$  by character  $\alpha$ .

STEP 2. There exists  $m' \in M \cap \tau^\perp$  such that  $z^{m+m'}$  is regular on  $B$ , i.e.,  $m + m' \in M \cap \sigma^\vee$ . Since the monomial  $z^m$  is  $K^*$ -semi-invariant, there exists an affine open  $K^*$ -invariant neighborhood of  $p$  on which  $z^m$  is regular, and since  $p \notin F_0^*$ , this neighborhood can be chosen of the form  $\pi^{-1}(V)$  where  $\pi : B \rightarrow B//K^*$  is the projection and  $V \subset B//K^*$  is an affine open toric subvariety. Let  $m' \in M \cap \sigma^\vee \cap a^\perp$  be such that  $V$  is the nonvanishing locus of the monomial  $z^{m'}$ . Then  $z^{m'}$  as a monomial on  $B$  is invertible at  $p$ , has  $K^*$ -character 0, and replacing  $m'$  by a multiple if necessary, we have that  $z^{m+m'}$  is regular on  $B$ .  $\square$

**3.2. The torifying property of the toric ideal.** Suppose  $X$  is a locally toric variety with a locally toric action of  $K^*$ , and  $D \subset X$  a divisor compatible with the locally toric structure, that means, at each point of  $X$  we can find a toric chart  $\eta_p : V_p \rightarrow X_p$  such that  $D \cap V_p$  is the inverse image of some toric divisor  $D_p \subset X_p \setminus T$ . In this situation we want to know if  $(X \setminus D) \subset X$  is a toroidal embedding on which  $K^*$  acts toroidally. Clearly it suffices to show that  $(X_p \setminus D_p) \subset X_p$  is a toroidal embedding with a toroidal  $K^*$  action for all toric charts for the  $K^*$  action. The following lemma gives several equivalent conditions for this.

**Lemma 3.2.1.** *Let  $X = X(\Sigma, N)$  be a toric variety,  $D \subset X \setminus T$  a divisor in  $X$ , and let  $K^*$  act on  $X$  as a one-parameter subgroup of the torus  $T$ , corresponding to a lattice point  $a \in N$ . Then the following are equivalent:*

1.  $X \setminus D \subset X$  is a toroidal embedding on which  $K^*$  acts toroidally.

2. For every affine open toric subvariety  $X_\sigma \subset X$  corresponding to a cone  $\sigma \in \Sigma$ , there exists a toric variety  $X_{\sigma'}$  with an action of  $K^*$  as a one-parameter subgroup of the torus  $T'$  such that we have a decomposition

$$X_\sigma \cong \mathbb{A}^k \times X_{\sigma'}$$

$$D \cong \mathbb{A}^k \times (X_{\sigma'} \setminus T'),$$

where the action of  $K^*$  on  $X_\sigma$  is a product of the action on  $X_{\sigma'}$  with the trivial action on  $\mathbb{A}^k$ .

3. For every cone  $\sigma = \langle v_1, \dots, v_m \rangle \in \Sigma$ , with  $v_1, \dots, v_k$  corresponding to the irreducible toric divisors not in  $D$ , we have a decomposition

$$\sigma \cong \langle v_1, \dots, v_k \rangle \times \langle v_{k+1}, \dots, v_m \rangle$$

$$N \cong N' \times N'',$$

where  $N' \subset N$  is the sublattice generated by  $v_1, \dots, v_k$ , and  $N'' \subset N$  is a complementary sublattice containing  $v_{k+1}, \dots, v_m$  as well as the point  $a$ .

4. For every cone  $\sigma = \langle v_1, \dots, v_m \rangle \in \Sigma$ , and every  $v_i$  corresponding to an irreducible toric divisor not in  $D$ , we have a decomposition

$$\sigma \cong \langle v_i \rangle \times \langle v_1, \dots, \widehat{v_i}, \dots, v_m \rangle$$

$$N \cong N_i \times N_{\widehat{i}},$$

where  $N_i \subset N$  is the sublattice generated by  $v_i$ , and  $N_{\widehat{i}} \subset N$  is a complementary sublattice containing  $v_j$ ,  $j \neq i$  as well as the point  $a$ .

5. For every affine open toric subvariety  $X_\sigma \subset X$  corresponding to a cone  $\sigma \in \Sigma$ , and every irreducible toric divisor  $E$  in  $X$  not in  $D$ , there exists a toric variety  $X_{\sigma'}$  with an action of  $K^*$  as a one-parameter subgroup of the torus  $T'$  such that we have a decomposition

$$X_\sigma \cong \mathbb{A}^1 \times X_{\sigma'}$$

$$E \cong \{0\} \times X_{\sigma'},$$

where the action of  $K^*$  on  $X_\sigma$  is a product of the action on  $X_{\sigma'}$  with the trivial action on  $\mathbb{A}^1$ .

*Remark.* We say that a divisor  $E$  as in condition 5 is removed from the toroidal structure of  $T \subset X$ .

**Proof.** The equivalences  $2 \Leftrightarrow 3$  and  $4 \Leftrightarrow 5$  are simply translations between toric varieties and the corresponding fans. The equivalence  $3 \Leftrightarrow 4$  follows easily from the combinatorics of cones. For  $2 \Rightarrow 1$ , we cover  $\mathbb{A}^k \times X_{\sigma'}$  with toroidal charts of the form  $\mathbb{G}_m^k \times X_{\sigma'}$ . The converse will not be used in this paper, and we leave it to the reader.  $\square$

As before, let  $B$  be a quasi-elementary cobordism, with a relatively affine, locally toric  $K^*$  action. We further assume that  $B$  is nonsingular. Choose  $c_1, \dots, c_\mu \in \mathbb{Z}$  a finite set of integers representing all characters of the  $G_p$ -action on the tangent space of  $B$  at  $p$  for all  $p \in B$ . Let

$$I = I_{c_1} \cdots I_{c_\mu}$$

be the product of the  $c_i$ -toric ideals, and let  $B^{tor} \rightarrow B$  be the normalized blowing up of  $B$  along  $I$ . Since  $I$  is  $K^*$ -equivariant, we can lift the action of  $K^*$  to  $B^{tor}$ . Denote by  $D \subset B^{tor}$  the total transform of the support of  $I$ , and  $U_{B^{tor}} = B^{tor} \setminus D$ .

We remark that such a set  $\{c_1, \dots, c_\mu\}$  can be found by covering the quasi-elementary cobordism  $B$  with a finite number of toric charts, and collecting all characters of the  $K^*$ -action on the coordinates of the toric varieties appearing in these charts. We are also allowed to enlarge the set of  $c_i$  - this will be utilized in the following section.

To understand the morphism  $B^{tor} \rightarrow B$  we use the following easy lemma:

**Lemma 3.2.2.** *For any  $c \in \mathbb{Z}$ , the ideal  $I_c$  is nonzero.*

**Proof.** This can be seen from the toric picture given in Lemma 3.1.5: the lattice point  $a$  is primitive (since the  $K^*$ -action is effective), and  $\pm a$  do not lie in the cone  $\sigma$  (since  $B_- \cap B_+$  is a nonempty open set), therefore the hyperplane  $(a, \cdot) = c$  contains lattice points in  $\sigma^\vee$ . Thus the set of  $f \in \mathcal{O}_{V,p}$  of  $G_p$ -character  $\alpha$  is nonempty.  $\square$

It follows that  $B^{tor}$ , being the normalized blowing up of the product  $I_{c_1} \cdots I_{c_\mu}$ , satisfies a universal property: it is the minimal normal modification of  $B$  such that the inverse image of  $I_{c_i}$  is principal for all

*i.* This implies that  $B^{tor}$  is canonically isomorphic to the normalization of the variety obtained from  $B$  by first blowing up  $I_{c_1}$ , then the inverse image of  $I_{c_2}$ , and so on.

**Proposition 3.2.3.** *The variety  $B^{tor}$  is a quasi-elementary cobordism, with  $(B^{tor})_+ = B^{tor} \times_B B_+$  and  $(B^{tor})_- = B^{tor} \times_B B_-$ . Moreover, the embedding  $U_{B^{tor}} \subset B^{tor}$  is toroidal and  $K^*$  acts toroidally on this embedding.*

**Definition 3.2.4.** We call  $I$  a *torific ideal* and  $B^{tor} \rightarrow B$  a *torific blowing up*.

**Proof.** Suppose  $B^{tor}$  is not quasi-elementary, that means, there exists a non-constant orbit with both of its limits in  $B^{tor}$ . Since  $B$  is quasi-elementary and the morphism  $B^{tor} \rightarrow B$  is equivariant, the image of this orbit must be a fixed point. However, the coordinate ring of an affine chart in a  $K^*$ -invariant fiber of the morphism  $B^{tor} \rightarrow B$  is generated by fractions  $f = f_1/f_2$  where  $f_i$  are generators of the ideal  $I$ , hence  $K^*$  acts trivially on  $f$ . This means that the fiber consists of fixed points, a contradiction.

Since  $(B^{tor})^{K^*}$  is the inverse image of  $B^{K^*}$ , we get that  $x \in (B^{tor})_+$  if and only if its image is in  $B_+$ , and similarly for  $(B^{tor})_-$ .

To prove that  $U_{B^{tor}} \subset B^{tor}$  is toroidal and  $K^*$  acts toroidally on this embedding, we consider toric charts  $\eta_p : V_p \rightarrow X_p$  in  $B$  giving the action of  $K^*$  on  $B$  a locally toric structure. By Lemma 3.1.2 the ideal  $I$  restricted to  $V_p$  is the inverse image of the ideal  $I_p = I_{p,c_1} \cdots I_{p,c_\mu}$  in  $X_p$ . It follows that the normalization of the blowing up of  $I_p$  in  $X_p$  provides a toric chart for  $B^{tor}$  such that the action of  $K^*$  on  $B^{tor}$  is again locally toric. Let  $X_q$  be such a chart:

$$\begin{array}{ccccc} X_q & \leftarrow & V_q & \subset & B^{tor} \\ \downarrow & & \downarrow & & \downarrow \\ X_p & \leftarrow & V_p & \subset & B. \end{array}$$

Let  $D_q \subset X_q$  be the support of the divisor defined by the total transform of  $I_p$ . Then

$$U_{B^{tor}} \cap V_q = \eta_q^{-1} X_q \setminus D_q,$$

and we are reduced to proving that  $(X_q \setminus D_q) \subset X_q$  is a toroidal embedding on which  $K^*$  acts toroidally. We do this by verifying the equivalent condition 5 in Lemma 3.2.1.

Let  $X_p = X(N, \sigma)$  be a nonsingular affine toric variety defined by the cone  $\sigma = \langle v_1, \dots, v_m \rangle$ ,  $\sigma^\vee = \langle v_1^*, \dots, v_m^*, \pm v_{m+1}^*, \dots, \pm v_n^* \rangle$ , and let  $K^*$  act on  $z_i$  by character  $c_i$ . The only irreducible toric divisors in  $X_q$  that do not lie in the total transform of  $I_p$  are among the strict transforms of the divisors  $\{z_i = z^{v_i^*} = 0\} \subset X_p$ . Consider the divisor  $\{z_1 = 0\}$ . The ideal  $I_{p,c_1}$  contains  $z_1$ . If  $I_{p,c_1}$  is principal then the strict transform of  $\{z_1 = 0\}$  is a component of  $D_q$ . Assume that this is not the case and choose monomial generators for  $I_{p,c_1}$  corresponding to lattice points  $v_1^*, m_1, \dots, m_l$  in  $M \cap \sigma^\vee$ . We may assume that  $m_i$  do not contain  $v_1^*$ , i.e., all  $m_i$  lie in the face  $v_1^\perp \cap \sigma^\vee = \langle v_2^*, \dots, \pm v_n^* \rangle$  of  $\sigma^\vee$ . To study the strict transform of  $\{z_1 = 0\}$  in  $X_q$  we first blow up  $I_{p,c_1}$ , then the rest of the  $I_{p,c_i}$ , and then normalize.

Let  $Y$  be an affine chart of the blowing up of  $X_p$  along  $I_{p,c_1}$  (not necessarily normal), obtained by inverting one of the generators of  $I_{p,c_1}$ , and let  $E$  be the strict transform of  $\{z_1 = 0\}$  in  $Y$ . Then  $E$  is nonempty if and only if  $Y$  is the chart of the blowing up where we invert one of the  $m_i$ , say  $m_1$ . Hence the coordinate ring of  $Y$  is generated by monomials corresponding to the lattice points

$$v_1^* - m_1, m_2 - m_1, \dots, m_l - m_1, v_2^*, \dots, \pm v_n^*.$$

Since the coefficient of  $v_1^*$  in  $v_1^* - m_1$  is 1, and the other generators lie in  $v_1^\perp$ , we have

$$\begin{aligned} Y &= \text{Spec } K \left[ \frac{z_1}{z^{m_1}}, \frac{z^{m_2}}{z^{m_1}}, \dots, \frac{z^{m_l}}{z^{m_1}}, z_2, \dots, z_n^{\pm 1} \right] \\ &= \text{Spec } K \left[ \frac{z_1}{z^{m_1}} \right] \times \text{Spec } K \left[ \frac{z^{m_2}}{z^{m_1}}, \dots, \frac{z^{m_l}}{z^{m_1}}, z_2, \dots, z_n^{\pm 1} \right] \\ &= \mathbb{A}^1 \times Y', \end{aligned}$$

where the strict transform  $E$  of  $\{z_1 = 0\}$  is defined by  $z_1/z^{m_1} = 0$ , on which  $K^*$  acts trivially.

It remains to be shown that if we blow up the ideals  $I_{p,c_i}$  for  $i \neq 1$  pulled back to  $Y$  and normalize, this product structure is preserved. We define the ideals  $I_{c_i}^Y$  on  $Y$  generated by all monomials on which  $K^*$  acts by character  $c_i$ . The lemma below shows that  $I_{c_i}^Y$  is equal to the inverse image of  $I_{p,c_i}$ . Hence we may blow up  $I_{c_i}^Y$  instead of the inverse image of  $I_{p,c_i}$ . Since  $K^*$  acts trivially on  $z_1/z^{m_1}$ , the ideals  $I_{c_i}^Y$  are generated

by monomials in the second term of the product. Thus, blowing up  $I_{c_i}^Y$  preserves the product, and so does normalization.  $\square$

**Lemma 3.2.5.** *For an affine toric variety  $X$  with an action of  $K^*$  as a one-parameter subgroup of the torus, let  $I_\alpha^X$  be the ideal generated by all monomials on which  $K^*$  acts by character  $\alpha$ . If  $\phi : Y \rightarrow X$  is a chart of the blowing up of  $I_\alpha^X$  then*

$$I_\beta^Y = (\phi^{-1} I_\beta^X) \mathcal{O}_Y$$

for all  $\beta$ .

**Proof.** Clearly  $\phi^{-1} I_\beta^X \subset I_\beta^Y$ . For the converse, let the monomial generators of the coordinate ring of  $Y$  be  $z_1/z_{m_1}, z^{m_2}/z^{m_1}, \dots, z^{m_l}/z^{m_1}, z_1, \dots, z_n^{\pm 1}$  for some generators  $z^{m_i}$  of  $I_\alpha$ . Thus a monomial on  $Y$  can be written as a product

$$z^m = \left(\frac{z_1}{z^{m_1}}\right)^{b_1} \left(\frac{z^{m_2}}{z^{m_1}}\right)^{b_2} \dots \left(\frac{z^{m_l}}{z^{m_1}}\right)^{b_l} \cdot z_1^{d_1} \dots z_n^{d_n}$$

for some integers  $b_i, d_j \geq 0$  for  $i = 1, \dots, l, j = 1, \dots, n$ . If  $z^m$  happens to be a generator of  $I_\beta^Y$ , i.e.,  $K^*$  acts on  $z^m$  by character  $\beta$ , then also  $K^*$  acts on  $z^{m'} = z_1^{d_1} \dots z_n^{d_n}$  by character  $\beta$ , and  $z^{m'}$  is in  $\phi^{-1} I_\beta^X$ .  $\square$

**Corollary 3.2.6.** *The embeddings  $U_{B_\pm^{tor}}/K^* \subset B_\pm^{tor}/K^*$  are toroidal embeddings, and the birational map  $B_-^{tor}/K^* \dashrightarrow B_+^{tor}/K^*$  is tightly toroidal.*

**Proof.** This is immediate from the proposition and Lemma 2.6.1.  $\square$

In fact, as the following lemma, in conjunction with 3.2.5, shows, the map  $B_-^{tor}/K^* \dashrightarrow B_+^{tor}/K^*$  is an isomorphism if the set  $\{c_1, \dots, c_\mu\}$  in the definition of the toric ideal  $I = I_{c_1} \dots I_{c_\mu}$  is chosen large enough. Since we do not need this result, we only give a sketch of the proof.

**Lemma 3.2.7.** *Let  $B = X(N, \sigma) = \text{Spec } K[z_1, \dots, z_n^{\pm 1}]$  be a nonsingular affine toric variety, and assume that  $K^*$  acts on  $z_i$  by character  $c_i$ . Let  $\alpha \in \mathbb{Z}$  be divisible by all  $c_i$ , and let  $I_\alpha$  and  $I_{-\alpha}$  be the ideals generated by all monomials of  $K^*$ -character  $\alpha$  and  $-\alpha$ , respectively. If  $\tilde{B}$  is the normalization of the blowing up of  $I_\alpha \cdot I_{-\alpha}$  then the birational map*

$$\tilde{B}_-/K^* \dashrightarrow \tilde{B}_+/K^*$$

*is an isomorphism. The same holds for any toric ideal corresponding to a set of characters containing  $\alpha$  and  $-\alpha$ .*

**Sketch of proof.** Let  $\sigma = \langle v_1, \dots, v_m \rangle$ , and let  $\pi : N_{\mathbb{R}} \rightarrow N_{\mathbb{R}}/\mathbb{R} \cdot a$  be the projection from  $a$ . If  $\pi$  maps  $\sigma$  isomorphically to  $\pi(\sigma)$  then  $B_-$  and  $B_+$  are isomorphic already. Otherwise, there exist unique rays  $r_+ \subset \partial_+ \sigma$  and  $r_- \subset \partial_- \sigma$  such that the star subdivision of  $\pi(\partial_+ \sigma)$  at  $\pi(r_+)$  is equal to the star subdivision of  $\pi(\partial_- \sigma)$  at  $\pi(r_-)$ . Now the normalized blowings up of  $I_\alpha$  and  $I_{-\alpha}$  turn out to correspond to star subdivisions of  $\sigma$  at  $r_+$  and  $r_-$ . The resulting subdivision  $\Sigma$  clearly satisfies  $\pi(\partial_- \Sigma) = \pi(\partial_+ \Sigma)$ .  $\square$

It is useful to have a more detailed description of the coordinate ring of some affine toric charts of  $B^{tor}$ . The strict transforms of the divisors  $\{z_i = 0\}$  corresponding to the ideals  $I_{c_i}$  which are not principal are removed from the toroidal structure on  $B^{tor}$ . Assume  $\tau$  is a cone in the subdivision associated to the normalization of the blowing up of a toric ideal, which contains  $v_1, \dots, v_k$ , the rays in  $\tau$  corresponding to the divisors to be removed from the toroidal structure. We have seen above that the corresponding affine toric variety  $Y$  decomposes as

$$Y = \text{Spec } K[z_i/z^{m_i}] \times Y'.$$

Since  $v_j \in \tau$  we have that  $(v_i^* - m_i, v_j) \geq 0$  for  $i, j = 1, \dots, k$ . Since  $m_i$  is positive we have

$$(m_i, v_j) = 0, \quad i, j = 1, \dots, k.$$

Note that we have a direct product decomposition of cones and lattices dual to the one in condition 4 of Lemma 3.2.1:

$$\begin{aligned} \tau^\vee &\cong \langle v_i^* - m_i \rangle \times \tau' \\ M &\cong M_i \times M_{\tau'}, \end{aligned}$$

Since, by condition 3 of Lemma 3.2.1, the direct product decompositions are compatible, we obtain the following:

**Corollary 3.2.8.** *Let  $B = X(N, \sigma) = \operatorname{Spec} K[z_1, \dots, z_n^{\pm 1}]$  be a nonsingular affine toric variety, and assume that  $K^*$  acts on  $z_i$  by character  $c_i$ . Let  $Y \subset B^{\text{tor}}$  be an affine toric chart corresponding to a cone  $\tau$  containing  $v_1, \dots, v_k$ , the rays in  $\tau$  corresponding to the divisors to be removed from the toroidal structure. Then there exist  $m_i \in \sigma^\vee$  such that  $(m_i, v_j) = 0$  for  $i, j = 1, \dots, k$  and  $z_i/z^{m_i}$  are invariant, and a toric variety  $Y'$ , such that*

$$Y = \operatorname{Spec} K \left[ \frac{z_1}{z^{m_1}}, \dots, \frac{z_k}{z^{m_k}} \right] \times Y'.$$

**Example 3.2.9.** Consider  $B = \mathbb{A}^3 = \operatorname{Spec} K[z_1, z_2, z_3]$ , where  $t \in K^*$  acts as

$$t \cdot (z_1, z_2, z_3) = (t^2 z_1, t^3 z_2, t^{-1} z_3).$$

We have the following generators of the toric ideals  $I_\alpha$ :

$$I_2 = \{z_1, z_2 z_3\}$$

$$I_3 = \{z_2, z_1^2 z_3\}$$

$$I_6 = \{z_1^3, z_2^2, z_1^2 z_2 z_3\}$$

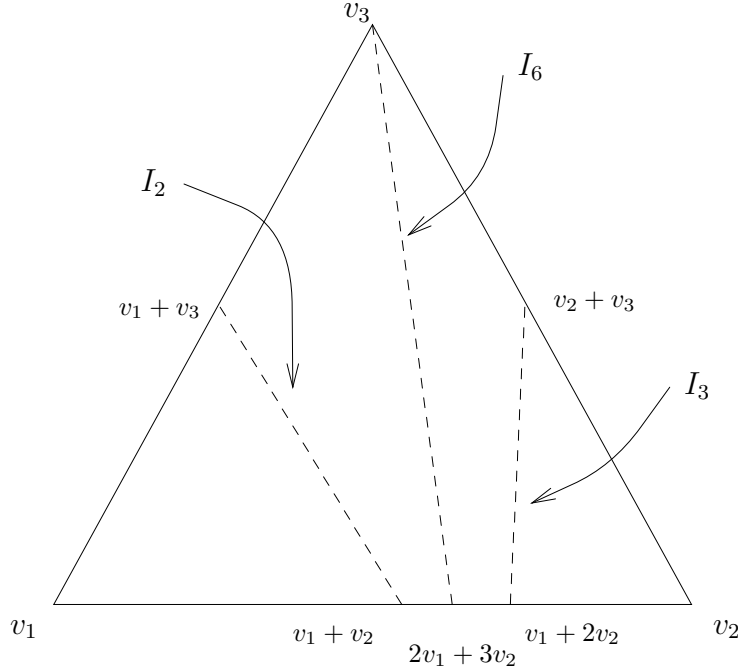
$$I_{-1} = \{z_3\}.$$

Let  $I = I_2 I_3 I_6 I_{-1}$ . If we regard  $B = X(N, \sigma)$  as the toric variety corresponding to the cone

$$\sigma = \langle v_1, v_2, v_3 \rangle \subset N_{\mathbb{R}},$$

then  $B^{\text{tor}}$  is described by the fan covered by the following four maximal cones

$$\begin{aligned} \sigma_1 &= \langle v_1, v_1 + v_3, v_1 + v_2 \rangle \\ \sigma_2 &= \langle v_1 + v_2, v_1 + v_3, 2v_1 + 3v_2, v_3 \rangle \\ \sigma_3 &= \langle 2v_1 + 3v_2, v_3, 2v_2 + v_1, v_1 + v_2, v_2 + v_3 \rangle. \\ \sigma_4 &= \langle 2v_2 + v_1, v_2 + v_3, v_2 \rangle \end{aligned}$$



The dual cone  $\sigma_1^\vee$  has the product description

$$\begin{aligned} \sigma_1^\vee &= \langle v_1^* - (v_2^* + v_3^*), v_2^*, v_3^* \rangle \\ &= \langle v_1^* - (v_2^* + v_3^*) \rangle \times \langle v_2^*, v_3^* \rangle. \end{aligned}$$

Thus, even if we remove the divisor  $\{z_1/z_2 z_3 = 0\}$  from the original toric structure of

$$X(N, \sigma_1) = \operatorname{Spec} k[z_1/z_2 z_3, z_2, z_3],$$

we still have the toroidal embedding structure

$$X(N, \sigma_1) \searrow (\{z_2 = 0\} \cup \{z_3 = 0\}) \subset X(N, \sigma_1).$$

As  $z_1/z_2z_3$  is invariant, the action of  $K^*$  is toroidal. For example, at  $0 \in X(N, \sigma_1)$  we have a toric chart

$$\begin{aligned} K^* \times K^2 &\rightarrow K \times K^2 \cong X(N, \sigma_1) \\ (x_1, x_2, x_3) &\mapsto (x_1 - 1, x_2, x_3). \end{aligned}$$

Globally, the divisors corresponding to the new rays

$$D_{\langle v_1+v_2 \rangle}, D_{\langle v_1+v_3 \rangle}, D_{\langle 2v_1+3v_2 \rangle}, D_{\langle v_1+2v_2 \rangle}, D_{\langle v_2+v_3 \rangle}$$

together with  $D_{\langle v_3 \rangle}$  coming from  $I_{-1}$ , are obtained through the blowing up of the toric ideals. Considering

$$U_{B^{tor}} = B^{tor} \searrow (D_{\langle v_1+v_2 \rangle} \cup D_{\langle v_1+v_3 \rangle} \cup D_{\langle 2v_1+3v_2 \rangle} \cup D_{\langle v_1+2v_2 \rangle} \cup D_{\langle v_2+v_3 \rangle} \cup D_{\langle v_3 \rangle})$$

we obtain a toroidal structure  $U_{B^{tor}} \subset B^{tor}$  with a toroidal  $K^*$ -action.

#### 4. A PROOF OF THE WEAK FACTORIZATION THEOREM

**4.1. The situation.** In Theorem 2.6.2 we have constructed a factorization of the given birational map  $\phi$  into tightly locally toric birational maps

$$\begin{array}{ccccccc} X_1 = W_{1-} & \dashrightarrow & W_{1+} \cong W_{2-} & \dashrightarrow & W_{2+} \cdots W_{m-} & \dashrightarrow & W_{m+} = X_2, \\ & \searrow & \swarrow & \searrow & \swarrow & \searrow & \swarrow \\ & B_{a_1} // K^* & & B_{a_2} // K^* & & B_{a_m} // K^* & \end{array}$$

where  $W_{i\pm} = (B_{a_i})_{\pm} / K^*$  (here  $W_{i-}$  is  $W_{i-1}$  in the notation of Theorem 2.6.2, and  $W_{i+}$  is  $W_i$ ).

For a choice of a toric ideal  $I = I_{c_1} \cdots I_{c_\mu}$  on  $B_{a_i}$ , denote by  $B_{a_i}^{tor} \rightarrow B_{a_i}$  the corresponding toric blowing up. Write  $W_{i\pm}^{tor} = B_{a_i\pm}^{tor} / K^*$ , and  $U_{i\pm}^{tor} = U_{B_{a_i\pm}^{tor}} / K^*$ . We have a natural diagram of birational maps

$$\begin{array}{ccccc} W_{i-}^{tor} & & \dashrightarrow & & W_{i+}^{tor} \\ \downarrow f_{i-} & \searrow & & \swarrow & \downarrow f_{i+} \\ W_{i-} & & B_{a_i}^{tor} // K^* & & W_{i+} \\ & \searrow & \downarrow & \swarrow & \\ & & B_{a_i} // K^* & & \end{array}$$

By Corollary 3.2.6 the embeddings  $U_{i\pm}^{tor} \subset W_{i\pm}^{tor}$  are toroidal, and the birational map  $\varphi_i^{tor} : W_{i-}^{tor} \dashrightarrow W_{i+}^{tor}$  is tightly toroidal.

We say that the ideal  $I = I_{c_1} \cdots I_{c_\mu}$  is *balanced* if  $\sum c_j = 0$ . It follows from lemma 3.2.2 that we can always enlarge the set  $\{c_1, \dots, c_\mu\}$  to get a balanced toric ideal  $I$ .

**Lemma 4.1.1.** *Suppose the toric ideal  $I$  is balanced. Then the morphism  $f_{i\pm}$  is a blowing up of a canonical ideal sheaf  $I_{i\pm}$  on  $W_{i\pm}$ .*

**Proof.** By Lemma 3.1.5, the ideal  $I$  is generated by  $K^*$ -invariant sections, and we can identify  $I$  as the inverse image of an ideal sheaf in  $B_{a_i} // K^*$  generated by the same sections. Let  $I_{i\pm}$  be the pullback of this ideal sheaf to  $(B_{a_i})_{\pm} / K^*$  via the map  $(B_{a_i})_{\pm} / K^* \rightarrow B_{a_i} // K^*$ . Then  $f_{i\pm}$  is the blowing up of  $I_{i\pm}$  because taking the quotient by  $K^*$  commutes with blowing up the sheaf  $I$ .  $\square$

We note that this lemma is true even when  $I$  is not balanced; however if  $I$  is not balanced the construction of a *canonical* ideal sheaf is less immediate. *From now on we assume that the toric ideals are chosen to be balanced.*

Note that if the varieties  $W_{i\pm}$  were nonsingular and the morphisms  $f_{i\pm}$  were composites of blowings up of smooth centers, we would get the weak factorization by applying Theorem 1.6.1 to each  $\varphi_i^{tor}$ . This is not the case in general. In this section we replace  $W_{i\pm}$  by nonsingular varieties and  $f_{i\pm}$  by composites of blowings up with nonsingular centers.

**4.2. Lifting toroidal structures.** Let  $W_{i\pm}^{res} \rightarrow W_{i\pm}$  be the canonical resolution of singularities. Note that, since  $W_{i+} = W_{(i+1)-}$ , we have  $W_{i+}^{res} = W_{(i+1)-}^{res}$ .

Denote  $I_{i\pm}^{res} = I_{i\pm} \mathcal{O}_{W_{i\pm}^{res}}$ . Let  $W_{i\pm}^{can} \rightarrow W_{i\pm}^{res}$  be the canonical principalization of the ideal  $I_{i\pm}^{res}$ , and let  $h_{i\pm} : W_{i\pm}^{can} \rightarrow W_{i\pm}^{tor}$  be the induced morphism.

$$\begin{array}{ccccc}
 W_{i-}^{can} & \xrightarrow{h_{i-}} & W_{i-}^{tor} & \dashrightarrow & W_{i+}^{tor} & \xleftarrow{h_{i+}} & W_{i+}^{can} \\
 \downarrow & & \downarrow f_{i-} & \searrow & \downarrow f_{i+} & & \downarrow \\
 W_{i-}^{res} & \rightarrow & W_{i-} & & W_{i+} & \leftarrow & W_{i+}^{res} \\
 & & \searrow & & \swarrow & & \\
 & & B_{a_i}^{tor} // K^* & & & & \\
 & & \downarrow & & & & \\
 & & B_{a_i} // K^* & & & & 
 \end{array}$$

Denote  $U_{i\pm}^{can} = h_{i\pm}^{-1} U_{i\pm}^{tor}$ . The crucial point now is to show:

**Proposition 4.2.1.** *The embedding  $U_{i\pm}^{can} \subset W_{i\pm}^{can}$  is a toroidal embedding, and the morphism  $W_{i\pm}^{can} \rightarrow W_{i\pm}^{tor}$  is toroidal.*

**Proof.** For simplicity of notation we drop the subscripts  $i$  and  $a_i$ , as we treat each quasi-elementary piece separately. We may assume that all the varieties  $B, W_{\pm}, W_{\pm}^{tor}, W_{\pm}^{res}, W_{\pm}^{can}$  and the morphisms between them are toric. Indeed, if  $V_p \rightarrow X_p$  is a toric chart at some point  $p \in W_{\pm}$ , obtained from a toric chart in  $B$ , we get a toric chart for  $W_{\pm}^{tor}$  by blowing up a toric ideal in  $X_p$ , which is a toric ideal since it is generated by monomials. Similarly, resolution of singularities and principalization over the toric variety  $X_p$  provide toric charts for  $W_{\pm}^{res}$  and  $W_{\pm}^{can}$ . The maps are toric (i.e., torus equivariant) by canonicity.

Consider now the diagram of toric morphisms between toric varieties and the corresponding diagram of fans:

$$\begin{array}{ccc}
 W_{\pm}^{can} & \rightarrow & W_{\pm}^{tor} \\
 \searrow & & \downarrow \\
 & & W_{\pm}
 \end{array}
 \qquad
 \begin{array}{ccc}
 \Sigma_{\pm}^{can} & \rightarrow & \Sigma_{\pm}^{tor} \\
 \searrow & & \downarrow \\
 & & \Sigma_{\pm}
 \end{array}$$

Let  $X_{\sigma} \subset W_{\pm}^{tor}$  be an affine open toric subvariety corresponding to a cone  $\sigma \in \Sigma_{\pm}^{tor}$ , and write

$$X_{\sigma} \cong \mathbb{A}^k \times X_{\sigma'},$$

where the toric divisors  $E_1, \dots, E_k$  pulled back from  $\mathbb{A}^k$  are the ones removed in order to define the toroidal structure on  $W_{\pm}^{tor}$ . Let  $X_{\sigma}^{can}$  be the inverse image of  $X_{\sigma}$  in  $W_{\pm}^{can}$ . We need to show that we have a decomposition  $X_{\sigma}^{can} \cong \mathbb{A}^k \times X_{\sigma'}^{can}$ , such that the resulting map  $\mathbb{A}^k \times X_{\sigma'}^{can} \rightarrow \mathbb{A}^k \times X_{\sigma'}$  is a product, with the second factor being the identity map.

Write  $X_{\sigma} = B_{\sigma}/K^*$ , where  $B_{\sigma} \subset B_{\pm}^{tor}$  is the affine open toric subvariety lying over  $X_{\sigma}$ .

By Corollary 3.2.8, the coordinate rings of  $B_{\sigma}$  and  $X_{\sigma}$  can be written as

$$\begin{aligned}
 A_{X_{\sigma}} &\cong K\left[\frac{z_1}{z^{m_1}}, \dots, \frac{z_k}{z^{m_k}}\right] \otimes A_{X_{\sigma'}}, \\
 A_{B_{\sigma}} &\cong K\left[\frac{z_1}{z^{m_1}}, \dots, \frac{z_k}{z^{m_k}}\right] \otimes A_{B_{\sigma'}},
 \end{aligned}$$

where  $X_{\sigma'} = B_{\sigma'}/K^*$ , and where  $z^{m_j}$  are monomials on which  $K^*$  acts with the same character as on  $z_j$ , such that  $z_i \nmid z^{m_j}$  for  $i, j = 1, \dots, k$ .

**Lemma 4.2.2.** *For each  $y = (y_1, \dots, y_k) \in K^k$  consider the automorphism  $\theta_y$  of  $B$  defined by*

$$\begin{aligned}
 \theta_y(z_i) &= z_i + y_i \cdot z^{m_i}, & i \leq k \\
 \theta_y(z_i) &= z_i, & i > k.
 \end{aligned}$$

*Then*

1.  $\theta_y$  defines an action of the additive group  $K^k$  on  $B$ .
2. The action of  $\theta_y$  commutes with the given  $K^*$ -action.
3. The ideals  $I_c$  are invariant under this action.
4. The action leaves  $B_{\pm}$  invariant, and descends to  $W_{\pm}$ .
5. The action lifts to  $B^{tor}$ .
6. This action on  $B^{tor}$  leaves the open set  $B_{\sigma}$  invariant.
7. The induced action on  $B_{\sigma}$  descends to a fixed-point-free action of  $K^k$  on  $X_{\sigma}$ .

8. The resulting action on  $X_\sigma$  is given by

$$\bar{\theta}_y(z_i/m_i) = z_i/m_i + y_i; \quad \theta_c(f) = f \text{ for } f \in A_{X_{\sigma'}}.$$

**Proof.** Since  $z_i \nmid m_j$  for  $i, j = 1, \dots, k$ , we have that the  $\theta_y$  commute with each other, and  $\theta_y + \theta_{y'} = \theta_{y+y'}$  thus defining a  $K^k$ -action. Since  $K^*$  acts on  $z_i$  and  $m_i$  through the same character, it commutes with  $\theta_y$ . For the same reason the ideals  $I_c$  are invariant. Since  $B_- = B \setminus V(\sum_{c < 0} I_c)$  we have that  $B_-$  is invariant, and similarly for  $B_+$ ; since the  $K^k$ -action commutes with  $K^*$  it descends to  $W_\pm$ . Since  $I = \prod I_{c_i}$  we have that  $I$  is  $K^k$ -invariant and therefore the  $K^k$ -action lifts to  $B^{\text{tor}}$ . Also by definition  $\theta_y(z_i/m_i) = z_i/m_i + y_i$ , which implies the rest of the statement.  $\square$

Back to the proposition. Since  $W_\pm^{\text{res}} \rightarrow W_\pm$  is the canonical resolution of singularities, the action of  $K^k$  lifts to  $W_\pm^{\text{res}}$ . Since the ideal  $I_\pm$  is generated by  $K^*$ -invariants in  $I$ , and since the action of  $K^*$  commutes with  $\theta_c$ , we have that  $I_\pm$  is invariant under  $K^k$ , and therefore  $I_\pm^{\text{res}}$  is invariant under  $K^k$  as well. Since  $W_\pm^{\text{can}} \rightarrow W_\pm^{\text{res}}$  is the canonical principalization of  $I_\pm^{\text{res}}$ , the action of  $K^k$  lifts to  $W_\pm^{\text{can}}$ . In particular, the map  $W_\pm^{\text{can}} \rightarrow W_\pm^{\text{tor}}$  is  $K^k$ -equivariant. By the lemma, the action of  $K^k$  on the invariant open set  $X_\sigma \subset W_\pm^{\text{tor}}$  is fixed-point free, therefore the action on the inverse image  $X_\sigma^{\text{can}}$  is fixed-point free. Writing  $X_{\sigma'}^{\text{can}}$  for the inverse image of  $(0, \dots, 0) \times X_{\sigma'}$ , we have an equivariant decomposition  $W_\pm^{\text{can}} \cong \mathbb{A}^k \times X_{\sigma'}^{\text{can}}$  as needed.  $\square$

**4.3. Conclusion of the proof.** Since  $X_1 = W_{1-}$  and  $X_2 = W_{m+}$  are nonsingular, we have  $W_{1-}^{\text{res}} = W_{1-}$  and  $W_{m+}^{\text{res}} = W_{m+}$ . For each  $i = 1, \dots, m$  we have obtained a diagram

$$\begin{array}{ccccc} & & W_{i-}^{\text{can}} \xrightarrow{\varphi_i^{\text{can}}} W_{i+}^{\text{can}} & & \\ & r_{i-} \swarrow & \downarrow h_{i-} \quad \downarrow h_{i+} & \searrow r_{i+} & \\ W_{i-}^{\text{res}} & & W_{i-}^{\text{tor}} \xrightarrow{\varphi_i^{\text{tor}}} W_{i+}^{\text{tor}} & & W_{i+}^{\text{res}} \\ \downarrow & f_{i-} \swarrow & & \searrow f_{i+} & \downarrow \\ W_{i-} & & \xrightarrow{\varphi_i} & & W_{i+} \end{array}$$

where

1. the canonical principalizations  $r_{i-}$  and  $r_{i+}$  are composites of blowings up with smooth centers,
2.  $\varphi_i^{\text{can}}$  is tightly toroidal.

Applying Theorem 2.7.1 to the toroidal map  $\varphi_i^{\text{can}}$  we see that  $\varphi_i^{\text{can}}$  is a composite of toroidal blowings up and blowings down, with smooth centers, between nonsingular toroidal embeddings. Thus we get a factorization

$$\phi : X_1 = W_{1-}^{\text{res}} \dashrightarrow W_{1+}^{\text{res}} = W_{2-}^{\text{res}} \dashrightarrow \dots \dashrightarrow W_{m-}^{\text{res}} \dashrightarrow W_{m+}^{\text{res}} = X_2,$$

where all  $W_i^{\text{res}}$  are nonsingular, and the birational maps are composed of a sequence of blowings up and blowings down. We do not touch the open subset  $U \subset X_1$  on which  $\phi$  is an isomorphism. After the reduction step in Lemma 1.3.1, the projectivity over  $X_2$  follows from Proposition 2.5.2, the projectivity statement in Theorem 2.7.1, and the construction. Finally, blowing up a nonsingular center can be factored as a sequence of blowings up of *irreducible* centers, simply blowing up one connected component at a time; since blowing up is a projective operation, this preserves projectivity. This completes the proof of Theorem 0.1.1.  $\square$

## 5. GENERALIZATIONS

**5.1. Reduction to an algebraically closed overfield.** We begin our proof of Theorem 0.3.1. We claim that, in case (1) of algebraic spaces, it suffices to prove the result in case  $L$  is algebraically closed. Let  $\bar{L}$  be an algebraically closed field containing  $L$ . Given  $\phi : X_1 \dashrightarrow X_2$ , isomorphic on  $U$ , consider the map  $\phi_{\bar{L}} : (X_1)_{\bar{L}} \dashrightarrow (X_2)_{\bar{L}}$ . Assuming the generalized factorization theorem applies over such a field, we get  $\varphi_{i\bar{L}} : \bar{V}_i \dashrightarrow \bar{V}_{i+1}$ . The functoriality of this factorization guarantees that the Galois group acts on  $\bar{V}_i$ , and  $\varphi_{i\bar{L}}$  are Galois equivariant. Therefore, denoting  $V_i = \bar{V}_i / \text{Gal}(\bar{L}/L)$ , we get  $\varphi_i : V_i \dashrightarrow V_{i+1}$  as required.



**5.2. Reduction to an algebraically closed subfield.** Still considering case (1), suppose  $L \subset K$  are algebraically closed fields, and suppose we have the theorem for algebraic spaces over fields isomorphic to  $L$ . If  $\phi : X_1 \dashrightarrow X_2$  is a birational map over  $L$ , with factorization given by  $\varphi_i : V_i \dashrightarrow V_{i+1}$ , then we claim that the induced maps  $\varphi_{iK} : V_{iK} \dashrightarrow V_{i+1K}$  is functorial over  $K$ . Indeed, any isomorphism  $K \rightarrow K'$  carries  $L$  to an isomorphic field, and the functoriality over  $L$  induces the desired morphisms  $V_{iK} \rightarrow V_{iK'}$ .

**5.3. Reduction to  $L = \mathbb{C}$ .** Still considering case (1), let  $K$  be algebraically closed and let  $\phi : X_1 \dashrightarrow X_2$  be a birational map of complete algebraic spaces over  $K$ . Then, by definition,  $X_i$  are given by étale equivalence relations  $R_i \subset Y_i^2$ , where  $R_i$  and  $Y_i$  are varieties over  $K$ , and  $\phi$  is defined by suitable correspondences between  $Y_i$ . Also the open set  $U$  corresponds to a Zariski open in  $Y_i$ . All these varieties can be defined over a finitely generated subfield  $L_0 \subset K$ , and therefore over its algebraic closure  $L \subset K$ . But any such  $L$  can be embedded in  $\mathbb{C}$ . Therefore, by the previous reductions, it suffices to consider the case of algebraic spaces over a field  $L$  isomorphic to  $\mathbb{C}$ .

By considering the associated analytic spaces, this allows us to use structures defined in the analytic category, as long as we ensure that the resulting blowings up are functorial in the algebraic sense, namely, independent of a choice of isomorphism  $L \rightarrow \mathbb{C}$ .

**5.4. Reduction to a projective morphism.** Now we consider both cases (1) and (2). To simplify the terminology, we use the term “birational map” to indicate also a bimeromorphic map. Given  $\phi : X_1 \dashrightarrow X_2$  isomorphic on  $U$ , let  $X'_i \rightarrow X_i$  be the canonical principalizations of  $X_i \setminus U$  (endowed with reduced structure). It is convenient to replace  $X_i$  by  $X'_i$  and assume from now on that  $X_i \setminus U$  is a simple normal crossings divisor.

We note that Lemma 1.3.1 works word for word in the cases of algebraic spaces or analytic spaces. As we have already remarked, this procedure is functorial. Also, the centers of blowing up have normal crossings with the inverse image of  $X_i \setminus U$ .

It is also easy to see that the resulting morphism  $X'_1 \rightarrow X'_2$  is endowed with a relatively ample line bundle which is functorial under absolute isomorphisms. Indeed, the Proj construction of a blowing up gives a functorial relatively ample line bundle for each blowing up. Furthermore, if  $f_1 : Y_1 \rightarrow Y_2$  and  $Y_2 \rightarrow Y_3$  are given relatively ample line bundles  $L_1$  and  $L_2$ , then there is a minimal positive integer  $k$  such that  $L_1 \otimes f_1^* L_2^{\otimes k}$  is relatively ample for  $Y_1 \rightarrow Y_3$ ; thus we can form a functorial relatively ample line bundle for a sequence of blowings up. In an analogous manner we can form a functorial ideal sheaf  $I$  on  $X'_2$  such that  $X'_1$  is the blowing up of  $I$ .

From now on we assume  $X_i \setminus U$  is a simple normal crossings divisor and  $\phi$  is a projective morphism.

**5.5. Analytic locally toric structures.** There are various settings in which one can generalize locally toric and toroidal structures to algebraic and analytic spaces, either using formal completions (see [34]), or étale charts (see [40]), or logarithmic structures (see [30]). Here we try to keep things simple, by sticking to the analytic situation, and modifying our earlier definitions slightly.

An *analytic toric chart*  $V_p \subset W$ ,  $\eta_p : V_p \rightarrow X_p$  is defined to be a neighborhood of  $p$  in the euclidean topology, with  $\eta_p$  an *open immersion* in the euclidean topology. The fact that we use open immersions simplifies our work significantly.

The notions of analytic locally toric structures, analytic toroidal embeddings, modifications, toroidal birational maps and tightly toroidal birational maps are defined as in the case of varieties, using analytic toric charts.

We note that in an analytic toroidal embedding, the toroidal divisors may have self intersections. If  $U \subset X$  is an analytic toroidal embedding, and if  $X' \rightarrow X$  is the canonical embedded resolution of singularities of  $X \setminus U$ , then  $X' \setminus U$  is a *strict toroidal embedding*, namely one without self intersections.

For strict toroidal embeddings, the arguments of [34] regarding rational conical polyhedral complexes, modifications and subdivisions go through, essentially word for word. The divisorial description of the cones (see [34], page 61) shows that the association  $(U \subset X) \mapsto \Delta_X$  of a polyhedral complex to a toroidal embedding is functorial under absolute isomorphisms in both the analytic and algebraic sense, and similarly for the modification associated to a subdivision.

**5.6. Functorial toroidal factorization.** Consider an analytic toroidal birational map  $\phi : W_1 \dashrightarrow W_2$  of complete nonsingular toroidal embeddings  $U \subset W_i$ . By the resolution of singularities argument above, we may assume  $U \subset W_i$  are strict toroidal embeddings. Theorem 2.7.1 applies in this situation, but we need to make the construction functorial. It may be appropriate to rewrite the proof in a functorial manner, but

this would take us beyond the intended scope of this paper. Instead we show here that the result can be made equivariant under the automorphism group of a fan cobordism, which, assuming the axiom of choice, implies functoriality.

Let  $\Delta_i$  be the polyhedral complex of  $U \subset W_i$ . Denote by  $G_i$  the automorphism group of  $\Delta_i$ . Since an automorphism of  $\Delta_i$  is determined by its action on the primitive points of the rays in  $\Delta_i$ , these groups are finite.

Consider the barycentric subdivision  $B\Delta_i \rightarrow \Delta_i$  (see [34], III 2.1, or [4]). It corresponds to a composition of blowings up  $BW_i \rightarrow W_i$ , which is functorial. The group  $G_i$  acts on  $B\Delta_i$ . The subdivision  $B\Delta_i \rightarrow \Delta_i$  has the following property: given a cone  $\sigma$  in  $B\Delta_i$ , an element  $g \in G_i$ , and a ray  $\tau$  in  $\sigma$  such that  $g\tau$  is also in  $\sigma$ , we have  $g\tau = \tau$ . This means, in particular, that for any subgroup  $H \subset G_i$  and any  $H$ -equivariant subdivision  $\Delta \rightarrow B\Delta_i$  the quotient  $\Delta/H$  is also a polyhedral complex (see [4]).

Let  $Z$  be the canonical resolution of singularities of the graph of  $BW_1 \dashrightarrow BW_2$ . This is clearly functorial in  $\phi$ . Now  $Z \rightarrow BW_i$  are toroidal birational morphisms, corresponding to subdivisions  $\Delta_Z \rightarrow B\Delta_i$ . Let  $H \subset G_1$  be the subgroup stabilizing the subdivision  $\Delta_Z \rightarrow B\Delta_1$ .

Fix a representative in the isomorphism class of  $\Delta_Z \rightarrow B\Delta_1$ , and, using the axiom of choice, fix an isomorphism of any element of the isomorphism class with this representative. Note that the absolute automorphism group of  $Z \rightarrow W_1$  maps to  $H$ . Therefore, in order to construct a functorial factorization of  $Z \rightarrow W_1$  it suffices to construct an  $H$ -equivariant combinatorial factorization of our representative of the isomorphism class, which by abuse of notation we call  $\Delta_Z \rightarrow B\Delta_1$ .

Now  $\Delta_Z/H \rightarrow B\Delta_1/H$  is a subdivision of nonsingular polyhedral complexes, and the toroidal weak factorization theorem says that it admits a combinatorial factorization, as a sequence composed of nonsingular star subdivisions and inverse nonsingular star subdivisions. Lifting these subdivisions to  $\Delta_Z \rightarrow B\Delta_1$ , we get the resulting  $H$ -equivariant factorization, which in turn corresponds to a toroidal factorization of  $BW_1 \dashrightarrow Z$ . We now apply the same procedure to  $Z' \rightarrow BW_2$ . This gives the desired functorial toroidal factorization of  $\phi$ .

**5.7. Analytic toroidal  $\mathbb{C}^*$ -actions.** The nature of  $\mathbb{C}^*$ -actions on analytic spaces differ significantly from the case of varieties. However, the situation is almost the same if one restricts to *relatively algebraic* actions.

**Definition 5.7.1.** Let  $X \rightarrow S$  be a morphism of analytic spaces and  $L$  a relatively ample line bundle for  $X \rightarrow S$ . An action of  $\mathbb{C}^*$  on  $X, L$  over  $S$  is *relatively algebraic* if there is an open covering  $S = \cup S_i$ , an algebraic action of  $\mathbb{C}^*$  on a projective space  $\mathbb{P}^{N_i}$ , and a Zariski-locally-closed  $\mathbb{C}^*$ -equivariant embedding  $X \times_S S_i \subset S_i \times \mathbb{P}^{N_i}$ , such that for some integer  $l_i$  we have that  $L_{X \times_S S_i}^{l_i}$  is  $\mathbb{C}^*$ -isomorphic to the pullback of  $\mathcal{O}_{\mathbb{P}^{N_i}}(1)$ .

It is easy to see that if  $X \rightarrow S$  is a projective morphism,  $L$  a line bundle, with a relatively algebraic  $\mathbb{C}^*$ -action, then  $X \subset \text{Proj}_S \text{Sym} E$ , where the sheaf  $E = \bigoplus_{i=1}^k E_i$  is a completely reducible  $\mathbb{C}^*$  sheaf.

In the analytic category we use embedded charts rather than étale ones. Accordingly, we say that a  $\mathbb{C}^*$ -equivariant open set  $V \subset X$  is *strongly embedded* if for any orbit  $O \subset V$ , the closure of  $O$  in  $X$  is contained in  $V$ . This implies that  $V//\mathbb{C}^* \rightarrow X//\mathbb{C}^*$  is an open embedding. We define an analytic locally toric  $\mathbb{C}^*$ -action on  $W$  using strongly embedded toric charts  $\eta_p : V_p \rightarrow X_p$  (we still have the requirement that  $V_p = \pi^{-1}\pi V_p$ , where  $\pi : W \rightarrow W//K^*$  is the projection, which means that  $V_p \subset W$  is also strongly embedded).

It is not difficult to show that a strongly embedded toric chart exists for each point  $p \in B$ , the analogue of Luna's fundamental lemma.

With these modification, Lemma 1.7.3 is proven in the same manner in the analytic setting. We also note that, if  $D = \sum_{i=1}^l D_i \subset W$  is a simple normal crossings divisor, then toric charts can be chosen compatible with  $D$ . Indeed, we only need to choose semi-invariant parameters  $x_1, \dots, x_n$  so that  $x_i$  is a defining equation for  $D_i$ , for  $i = 1, \dots, l$ .

**5.8. Analytic birational cobordisms.** Analytic birational cobordisms are defined the same way as in the case of varieties, with the extra assumption that the  $\mathbb{C}^*$ -action is relatively algebraic.

Given a projective birational morphism  $\phi : X_1 \rightarrow X_2$  we construct a compactified, relatively projective cobordism  $\overline{B} \rightarrow X_2$  as in the algebraic situation, with the following modification: using canonical resolution of singularities we make the inverse image of  $X_2 \setminus U$  in  $\overline{B}$  into a simple normal crossings divisor, crossing  $X_1$  and  $X_2$  normally. Note that these operations are functorial in absolute isomorphisms of  $\phi$ .

As indicated before, this construction endows  $\overline{B} \rightarrow X_2$  with a functorial relatively ample line bundle. Since this bundle is obtained from the Proj construction of the blowing up of an invariant ideal, it comes with a functorial  $\mathbb{C}^*$ -action as well.

The considerations of collapsibility and geometric invariant theory work as in the algebraic setting, leading to Theorem 2.6.2. We note that the resulting locally toric factorization is functorial, and the toric charts on  $W_i$  can be chosen compatible with the divisor coming from  $X_1 \searrow U$  or  $X_2 \searrow U$ .

**5.9. Functoriality of torification and compatibility with divisors.** We note that the definition of the torific ideals is clearly functorial. The proof of its existence works as in the case of varieties. The same is true for its torifying property. In order to make this construction compatible with divisors, we replace the total transform  $D$  of  $I$  by adding the inverse image of  $X_2 \searrow U$ . This guarantees that the resulting toroidal structure on  $B^{tor}$  is compatible with the divisors coming from  $X_2 \searrow U$ .

**5.10. Conclusion of proof of Theorem 0.3.1.** Canonical resolution of singularities is functorial, therefore the construction of  $W_{\pm}^{res} \rightarrow W_{\pm}$  is functorial. We can now replace  $W_{\pm}^{res}$  by the canonical principalization of the inverse image of  $X_2 \searrow U$ , making the latter a *simple* normal crossings divisor. Since the ideal  $I$  is functorial, the construction of  $W_{\pm}^{can} \rightarrow W_{\pm}$  is functorial, and the locally toric structure implies that the centers of blowing up in  $W_{\pm}^{can} \rightarrow W_{\pm}^{res}$  have normal crossings with the inverse image of  $X_2 \searrow U$ . We can now apply functorial toroidal factorization to the toroidal birational map  $W_{\pm}^{can} \dashrightarrow W_{\pm}^{can}$ . Note that the centers of blowing up, being toroidal, automatically have normal crossings with  $W_{\pm}^{can} \searrow U_{\pm}^{can}$ . The theorem follows.  $\square$

## 6. PROBLEMS RELATED TO WEAK FACTORIZATION

**6.1. Strong factorization.** Despite our attempts, we have not been able to use the methods of this paper to prove the strong factorization conjecture, even assuming the toroidal case holds true.

In the construction of the torific ideal in 3.1 and the analysis of its blowing up in 3.2 and 4.2, the assumption of the cobordism  $B_{a_i}$  being quasi-elementary is essential. One can extend the ideal over the entire cobordism  $B$ , for instance by taking the Zariski closure of its zero scheme, but the behavior of this extension (as well as others we have considered) along  $B \searrow B_{a_i}$  is problematic.

The weak factorization theorem reduces the strong factorization conjecture to the following problem:

**Problem 6.1.1.** *Let  $X_1 \rightarrow X_2 \rightarrow \cdots \rightarrow X_n$  be a sequence of blowings up with nonsingular centers, with  $X_n$  nonsingular, and such that the center of blowing up of  $X_i \rightarrow X_{i+1}$  has normal crossings with the exceptional divisor of  $X_{i+1} \rightarrow X_n$ . Let  $Y \rightarrow X_n$  be a blowing up with nonsingular center. Find a strong factorization of the birational map  $X_1 \dashrightarrow Y$ .*

We believe that at least the threefold case of this problem is tractable.

### 6.2. Toroidalization.

**Problem 6.2.1** (Toroidalization). *Let  $\phi : X \rightarrow Y$  be a surjective proper morphism between complete nonsingular varieties over an algebraically closed field of characteristic 0. Do there exist sequences of blowings up with smooth centers  $\nu_X : \tilde{X} \rightarrow X$  and  $\nu_Y : \tilde{Y} \rightarrow Y$  so that the induced map  $\tilde{\phi} : \tilde{X} \dashrightarrow \tilde{Y}$  is a toroidal morphism? Can such maps be chosen in a functorial manner, and in such a way that they preserve any open set where  $\phi$  admits a toroidal structure?*

A similar conjecture was proposed in [35]. We note that the toroidalization conjecture concerns not only birational morphisms  $\phi$  but also generically finite morphisms or morphisms with  $\dim X > \dim Y$ . The solution to the above conjecture would reduce the strong factorization conjecture to the toroidal case, simply by considering the case of a birational morphism  $\phi$  and then applying the toroidal case to  $\tilde{\phi}$ . At present the authors know a complete proof only if either  $\dim X = 2$  (see below), or  $\dim Y = 1$  (which follows immediately from resolution of singularities, see [34], II §3). Recently, S. D. Cutkosky announced a solution of the case  $\dim X = 3, \dim Y = 2$ .

The conjecture is false in positive characteristics due to wild ramifications. See, e.g., [14].

One general result which we do know is the following.

**Theorem 6.2.2.** *Let  $\phi : X \rightarrow Y$  be a proper surjective morphism between complete varieties over an algebraically closed field of characteristic 0. Then there exists a modification  $\nu_X : \tilde{X} \rightarrow X$  and a sequence of blowings up with smooth centers  $\nu_Y : \tilde{Y} \rightarrow Y$  so that the induced map  $\tilde{\phi} : \tilde{X} \dashrightarrow \tilde{Y}$  is a toroidal morphism.*

**Proof.** In [2], Theorem 2.1, it is shown that modifications  $\nu_X$  and  $\nu_Y$  such that  $\tilde{\phi}$  is toroidal exist, assuming  $X$  and  $Y$  are projective and the generic fiber of  $\phi$  is geometrically integral. We can reduce to the projective case using Chow's lemma. The case where the generic fiber is not geometrically integral is resolved in the second author's thesis [29]. Since the latter is not widely available we give a similar argument here. The inductive proof of [2], Theorem 2.1 reduces the problem to the case where  $\phi$  is generically finite. By Hironaka's flattening (or by taking a resolution of the graph of  $Y \dashrightarrow \text{Hilb}_Y(X)$ ), we may assume that  $X \rightarrow Y$  is finite. Using resolution of singularities, we may assume  $Y$  is nonsingular and the branch locus is a normal crossings divisor. By normalizing  $X$  we may assume  $X$  normal. Denoting the complement of the branch locus by  $U_Y$  and its inverse image in  $X$  by  $U_X$ , Abhyankar's lemma says that  $U_X \subset X$  is a toroidal embedding and  $X \rightarrow Y$  is toroidal, which is what we needed.

It remains to be shown that  $\nu_Y$  can be chosen to be a sequence of blowings up with nonsingular centers. Let  $Y \leftarrow Y' \rightarrow \tilde{Y}$  be a resolution of indeterminacies of  $Y \dashrightarrow \tilde{Y}$  and let  $Y'' \rightarrow Y'$  be the canonical principalization of the pullback of the ideal of the toroidal divisor of  $\tilde{Y}$ . Let  $X'' \rightarrow Y'' \times_{\tilde{Y}} \tilde{X}$  be the normalization of the dominant component. Then  $Y'' \rightarrow Y$  is a sequence of blowings up with nonsingular centers. Applying [2], Lemma 6.2, we see that  $X'' \rightarrow Y''$  is still toroidal, which is what we needed.  $\square$

Since every proper birational morphism of nonsingular surfaces factors as a sequence of point blowings up, we get:

**Corollary 6.2.3.** *The toroidalization conjecture holds for a generically finite morphism  $\phi : X \rightarrow Y$  of surfaces.*

In this case, it is not difficult to deduce that there exists a *minimal* toroidalization (since the configuration of intermediate blowings up in  $\tilde{X} \rightarrow X$  or  $\tilde{Y} \rightarrow Y$  forms a tree). This result has been proven in an algorithmic manner by Cutkosky and Piltant [14]. Similar statements can be found in [6].

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DEPARTMENT OF MATHEMATICS, BOSTON UNIVERSITY, 111 CUMMINGTON STREET, BOSTON, MA 02215, USA  
*E-mail address:* [abrmovic@math.bu.edu](mailto:abrmovic@math.bu.edu)

DEPARTMENT OF MATHEMATICS, HARVARD UNIVERSITY, ONE OXFORD STREET, CAMBRIDGE, MA 02139, USA  
*E-mail address:* [kkaru@math.harvard.edu](mailto:kkaru@math.harvard.edu)

DEPARTMENT OF MATHEMATICS, PURDUE UNIVERSITY, 1395 MATHEMATICAL SCIENCES BUILDING, WEST LAFAYETTE, IN 47907-1395, USA  
*E-mail address:* [kmatsuki@math.purdue.edu](mailto:kmatsuki@math.purdue.edu)

INSTYTUT MATEMATYKI UW, BANACHA 2, 02-097 WARSZAWA, POLAND  
*E-mail address:* [jwlodar@mimuw.edu.pl](mailto:jwlodar@mimuw.edu.pl)